THE UNIVERSITY OF ALBERTA

CANTOR'S "NEW WAY OF SEEING": A STUDY OF MATHEMATICAL CONVENTION

by

(C) WANDA JEAN TEAYS

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF ARTS

DEPARTMENT OF PHILOSOPHY

EDMONTON, ALBERTA

SPRING, 1972

THE UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled Cantor's "New Way of Seeing": A Study of Mathematical Convention, submitted by Wanda Jean Teays in partial fulfilment of the requirements for the degree of Master of Arts.

TABLE OF CONTENTS

INTRODUCTION

> It is possible of course to operate with figures mechanically, just as it is possible to speak like a parrot: but that hardly deserves the name of thought. It only becomes possible at all after the mathematical notation has, as a result of genuine thought, been so developed that it does the thinking for us, so to speak.[1]

Mathematical notation, like other mathematical conventions which have been developed, may be said to do 'the thinking for us'. In fact, it may be doing so much of the 'thinking' that any deviance from the habitual mode can be a real disputable issue.

The power of convention is underestimated -- simply because certain 'habits of thinking' have existed for so long that there is little awareness of their strength and rigidity. That is, a tradition can become so accepted that to think differently can be a major achievement in itself. As in most cases, this is certainly true in mathematics. A primary example -- and one which shall be focussed upon in this paper -- is the centuries' old acceptance of the 'potential' infinite as the only meaningful conception of 'infinity'. Probably the first concern with the 'potential' infinite, specifically its paradoxocal characteristics, was that of Zeno of Elea in his consideration of the two paradoxes, called, 'Achilles and the Tortoise' and 'The Dichotomy'.[2] These paradoxes puzzled mathematicians for

---

[1]Gottleib Frege, Foundations of Arithmetic, Basil Blackwell, Oxford, 1959; p.iv.

[2]The first mentioned paradox involves a race between Achilles, a well-known fast runner, and the Tortoise, a much slower one -- Because of the seeming differences in the running abilities of the two, the Tortoise was given a head-start in the race. If Achilles begins at location t_o and the Tortoise at a location T_o, then when Achilles has reached location T_o, the Tortoise shall have moved ahead to a new location, say T_1. While Achilles has covered this distance, and arrived at the point T_1, again the Tortoise shall have moved forward to another position, say T_2. And so on, throughout the race. Achilles, as a result, will never overtake the Tortoise.

The second paradox, again deals with the runner, Achilles, who is attempting to run a finite distance. However, Achilles can never cover the entire distance, since he must first run half the length, then half the remaining distance, then half the yet remaining distance, and so on. But since the finite distance is infinitely divisible by two, Achilles can never complete the distance, no matter how large or small.

centuries as they sought to resolve the dilemma. Further, the paradoxes of Zeno presented, very concretely, the notion of the 'potential' infinite -- for both paradoxes suggest the view of the infitesimal in the same manner as that of the 'potential' infinite, that is, as a process which has no completion, and, hence can never be seen as a finished totality.

What is significant to realize with regard to Zeno's paradoxes is, on one hand, the concern shown them, insofar as the paradoxes seem to demonstrate the bizarre, or anti-common sense, qualities of the concept (when we 'know' of course, that Achilles shall complete the race and beat the Tortoise). On the other hand, it is noteworthy that a resolution of the paradoxes seems to be obtainable in light of methods and concepts brought to fore in Cantorian set theory.[3]

From the time of Aristotle to that of Gauss and Kronecker in the 19th Century, the rarely questioned position on 'infinity' was that of a 'facon de parler', a symbol or limit which can never be reached or understood per se. Since it was being viewed as a 'potential', there was not much exposition on the concept. Consequently, the entire notion of 'infinity' continued to be (relatively) undeveloped until the 1890s, when a new direction was introduced by Georg Cantor. Cantor offered a new approach on the topic with his notion of the 'actual' infinite!

---

[3]For example, Kline argues (p.403 Mathematics in Western Culture) as follows: "We must agree that from the start of the race to the end the tortoise passes through as many points as Achilles does, because at each instant of time during which they run each occupies exactly one position. Hence, there is a 1-1 correspondence between the infinite set of points run through by the tortoise and the infinite set of points run through by Achilles. The assertion that because he must travel a greater distance to win the race Achilles will have to pass through more points than the tortoise is not correct, however, because, as we now know, the number of points on the line segment Achilles must traverse to win the race is the same as the number of points on the segment the tortoise traverses. Again we must notice that the number of points on a line segment has nothing to do with its length. In other words, it is Cantor's theory of infinite classes that solves the problem. . ."

Not only did such a creation astound the mathematical world with its radical deviation from the traditional viewpoint of the infinite, but Cantor's work on transfinite number theory revolutionized all of mathematics. The importance of Cantor's work cannot be underestimated, for his work had significance throughout both mathematics and philosophy.

Both the traditional view of the 'potential' infinite and Cantor's introduction of the 'actual' infinite shall be examined. An attempt will be made to demonstrate, not only the differences and contrasts between the two approaches on 'infinity', but, furthermore, I shall try to show that they are not necessarily conflicting. Instead, they are just two differing perspectives on 'infinity', rather than it being the case that one is 'right' and one 'wrong'.

In addition to the two notions on the infinite, certain important conventions within mathematics shall be examined, particularly since mathematical tradition had its effect on both Cantor himself and the reception which his creation received in the mathematical arena.

I shall discuss pertinent conventions and the previous view(s) on 'infinity', in order to determine the extent to which Cantor's theory did, in fact, offer a 'new way of seeing'. I hope to show that, although Cantor was quite influenced by mathematical conventions in general, he did offer a new (and thought provoking) direction on the concept of the infinite.

CHAPTER I

MATHEMATICAL CONVENTIONS

> The laws of logic are indeed the expression of 'thinking habits' but also of the habit of thinking. That is to say they can be said to shew: how human beings think, and also what human beings call 'thinking'.[1]

Within the sphere of 'thinking habits' and 'habits of thinking' is found mathematical thinking. And like the laws of logic which show what humans call 'thinking', the conventions or 'laws' of mathematics express mathematical thinking habits and show us what mathematicians call 'thinking mathematically'. That such thinking should differ from 'thinking' is seen only by understanding what kinds of 'laws' of mathematics exist. Only by a general overview of these habits and laws can one attempt to understand the rigidity of mathematics. Only by such an examination can one become aware of the demands which must be met before a new entity is accepted as 'mathematical'. Only then can one perceive the significance of the response of mathematicians as the new theories and creations of such mathematicians as Gauss, Lobachevsky, Riemann, Cantor became known throughout the mathematical arena. Only then can one determine the freedom of mathematical development and what conditions, if any, are necessary for deciding the acceptability of a mathematical creation. And only then can one understand the power of mathematical conventions and why a renowned mathematician like Gauss would not publish his new non-Euclidean theories during his lifetime and why a mathematical genius such as Cantor came to wonder if what he had created was mathematics at all.

Within mathematics, there are 'laws' about proof procedure, 'laws' regarding the way(s) in which diagrams and proofs should be 'seen', 'laws' as to the role and purpose of mathematics and mathematical theories, 'laws' regarding the words used in talking in and

---

[1]Ludwig Wittgenstein, Remarks on the Foundations of Mathematics, M.I.T. Press, Massachusetts; pp. I-41.

about mathematics, 'laws' within 'laws' and conventions within conventions. What must be sought is an understanding of the existence of such conventions and what their basic significance is.

One of the primary conventions one encounters when one first tries to understand what constitutes 'mathematical thinking' is related to proof procedure. There are 'laws' whereby one can determine what kind of proof is being presented; such as by induction, reductio ad adsurdum, proofs by reference (to a different theorem, corollary, or axiom), and so on. According to the perspective of each different mathematical system or viewpoint, there are certain styles of proofs which are or are not allowed on the basis of prevalent contentions and conventions. Furthermore, the mathematical conventions of a certain system allow only that the proof procedure be read in a more or less designated manner. In order to clarify one of the ways in which this is done, let us look at an example from the institutionistic school of mathematics. In this system there are certain types of transfinite inductive proofs and reductio ad adsurdum proofs -- namely those deriving a contradiction from the conclusion 'for all n, not P(n)' -- which are not considered permissible. And a proof by induction is one which must be performed and/or read in a certain way: i.e. using the inductive procedure 'for all n . . .', the institutionist would want it to be read as follows: When we speak of 'for all n' this is to mean that for any n, one could find, a property, say P(n), would hold. That is the 'constructible' method is that which the institutionists would designate as most desirable as a proof procedure.

That there should exist any kind of stringent 'laws' regarding proof procedure in a mathematical system, is indicative enough of the presence of habits and conventions. "Moreover, to call several pages of printed marks a proof presupposes a good deal of sociological circumstances which make them a proof".[2] One of the things to which this points is the abilities and habits which one must acquire in order

---

[2]Hao Wang, Process and Existence in Mathematics, from Bar-Hillel, Poznanski, Rabin, and Robinson (eds.), Essays on the Foundations of Mathematics, North-Holland Puc., 1962; p.337.

to even read something as a proof at all. To suppose that one can see something as a mathematical proof, there is a great deal that has to be taken for granted about the ability of the mind to abstract, group objects, and, in Dedekind's words, ". . . to let a thing correspond to a thing, or to represent a thing by a thing".

Quine seemed aware of this assumption (of the mind's ability to abstract) in his work on set theory. In an effort to describe without defining a 'set' or 'class', he stated:

> The notion of class is so fundamental to thought that we cannot hope to define it in more fundamental terms. We can say that a class is any aggregate, and collection, any combination of objects of any sort; if this helps, well and good. But even this will be less help than hindrance unless we keep clearly in mind that the aggregating or collecting or combining here is to connote no actual displacement of the objects, and further that the aggregation or collection of shoes is not to be identified with the aggregation or combination of those fourteen shoes.[3]

That is, one has to be able to group the shoes in such a prescribed manner, that one can 'see' the inequality between the set of seven pairs and the set of the fourteen shoes. And in order to achieve that, there is a good deal of mathematical training and learning of conventions that should be presupposed.

There is much significance surrounding the notion of proof in mathematics -- so much so, in fact, that an otherwise 'mathematical' system which was totally devoid of proofs (e.g. only containing 'postulates') would not be considered mathematics at all. Generally, or conventionally, speaking, a mathematicological statement or proposition for which there did not exist a proof is either meaningless, undecidable, irrelevant, false, a definition or a 'postulate' -- i.e. something which is utilized, though not proven, as 'true'.

If,

> The aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another[4]

---

[3]W.V.Quine, Set Theory and Its Logic, Belknap Press, Massachusetts, 1963; p.1.

[4]G. Frege, Foundations of Arithmetic, Basil Blackwell, Oxford, 1959; p.2.

then one finds the interesting convention that a mathematical proof establishes a mathematical truth and, further, that proofs are more highly valued the more they enable one to see into mathematical dependence. This convention helps one to understand the fear and disdain that mathematicians hold toward contradictory theories or statements. If a theory leads to paradoxes and contradictions, there is obviously going to be a gap between the theory or proposition and 'mathematical truths'. For instance, witness the response among mathematicians as they became increasingly aware of the contradictions and paradoxes associated with Cantorian set theory. And even though there were people such as Godel who believed "the set-theoretical paradoxes . . . are a very serious problem, not for mathematics however, but rather for logic and epistemology", it nevertheless was the case that most mathematicians were disdainful and/or distrustful of any theory, such as Cantor's, which ran into many paradoxes and contradictions.

It would seem that this leads to a kind of value system in mathematics[5] -- for by mathematical ethic the proof establishes the proposition as a 'truth'. And this ethic may help explain Wittgenstein's statement on the 'superstitious fear and awe of mathematicians in the face of contradiction', for it is then that the mathematician encounters uncertainty. In such a situation, there is a conflict of mathematical 'truths' and there are no clear basic prescriptions or conventions as to what one can do to resolve the conflict.

As was suggested, the Frege quote mentions the proof convention that their value is increased according to the insight given into the dependence (interdependence) of proofs, or, as he states, 'truths'. He suggests that such 'insight' be deemed one of the aims of proof, so that a mathematical system consists of notions which are very much dependent upon one another and the more a proof allows one to see that, the more successfully the proof has achieved its purpose. It would seem to demonstrate the homogeneity, or integration, of mathematics; <u>i.e.</u> that a mathematical theory is not at all a body of independent

---

[5]This is not to mention the mathematical ethic which utilizes such criteria as elegance, insight, and validity to determine which is the better of two proofs.

'truths', as might be thought. What might be interpreted from Frege's statement about the dependence of truths upon one another is that the 'truths' are dependent in that they are context-dependent. Namely, if a proof establishes a mathematical 'truth' then one might encounter, as in the case of the Euclidean and non-Euclidean geometries, the situation of conflicting or contradictory, 'truths'; since some of the theorems in some branches contradict (or conflict) some of the theorems in others. Hence unless we allow for the contextual dependence of 'truths', we will find ourselves faced with 'truths' in opposition to one another and, consequently, no 'truths' at all (in those circumstances). In that sense, whether or not there are others, one must recognize the contextual dependence of mathematical proofs ('truths').

Overlapping conventions related to proof procedure and mathematical establishment of 'truths' are those relating to 'seeing'. The 'seeing' of a proof or diagram is a learned art. That is, one is taught how to look at a proof and what order and manner in which a proof should be 'read'. One learns how to see one group of letters as a word, one group as symbols, one group as operations, and another group as a combination of any or all of symbols, operations, and words. One learns to see some letters as elements, some letters as classes or sets which the elements are members of. One can get these so learned, that, when glancing say at a group of letters on a page, one knows, for instance, that it is in Polish notation and thereby calls for certain procedures to determine what the letters indicate.

It is the 'seeing' of diagrammatic figures which illustrates this point more fully. That something is even seen as a diagram of mathematics is something which is a learned convention. Furthermore, that one should see some lines as two or three dimensional figures becomes not only a conventional, but also a contextual matter. To see something as a figure, in foreground or background, plane or solid, and so on, is a matter almost entirely based upon conventions. And it is a matter which is most obvious in the case of three dimensional objects because of the increasing complexities of perceiving the

figure(s) in such and such a way. And the ability to see something as a two or three dimensional figure, depending upon the mathematical context, is surely another evidence of the convention. One must realize, too, that this 'seeing' is learned very early in mathematical training and is something which becomes so inbred into one's mathematical 'thinking habits' as to be forgotten, as far as conscious awareness indicates.

What one encounters when investigating the extent of mathematical thinking and seeing 'habits' is the presence of a very rigid structure which surrounds mathematics -- and it surrounds it in such a way that viewing, reading, utilizing, and even talking about it becomes quite regimented. This points to yet another mathematical convention; namely, mathematical language. Even the language used to talk <u>about</u> a mathematical theory is shrouded in conventions -- and that is not even taking into account the various mathematical words and definitions. Here we find the mathematician's glossary: Not only are there various conventional meanings of mathematical words; but so too are there conventional meanings of words used in mathematics that are also found in every-day discourse. For instance, within the mathematical sphere alone,

> The words 'can', 'decidable', etc. mean different things in pure mathematics and applied mathematics, in actual mathematical activities and in the discussions of mathematical logicians.[6]

Consequently, one sees that there exists a whole kind of sub-language(s) within the language of mathematics, and the sub-language(s) are relative to the perspective or mathematical school of the theory's author and context.

Furthermore, there are 'prescriptions' about the usage of such 'mathematical' words as 'such that', 'proof', 'quod est demonstratum' (Q.E.D.), 'all', 'some', 'every', 'universe', 'universal', 'existential', 'set', 'type', 'finite', 'infinite', etc. Although there is a kind of

---

[6]Hao Wang, <u>Process and Existence in Mathematics</u>, from Poznanski Bar-Hillel, Rabin, and Robinson (eds.), <u>Essays on the Foundations of Mathematics</u>, North-Holland Puc., 1962; p.338.

general usage of these words, and others found when speaking 'mathematics' or about mathematics, where the conventions become most apparent is within the different mathematical perspectives -- that is where the relativity is exposed. To an institutionist, a 'there exists' would only be meaningful as a 'we can find', whereas to a formalist a 'there exists' would mean just that; that is, it wouldn't have to have such a strict constructible definition in order for the phrase to be meaningful in a mathematical proposition.

Zermelo speaks of the relativity of the word-convention in mathematics. And although his reference is to yet another mathematical phrase, the significance of his statement should be understood and related to almost any mathematical word or words one could name. As Zermelo contended:

> 'Finitely definable' is not an absolute notion but only a relative one, and it is always related to the chosen language or notation. The conclusion that (the set of) all finitely definable objects must be denumerable holds only if one and the same system of signs is to be used for all of them, and the question whether a single individual can or cannot have a finite designation is in and of itself meaningless, since to any object we could, if necessary, arbitrarily assign any designation whatever.[7]

Acknowledging the arbitrariness of mathematical language and sub-languages, it would consequently appear that a language which would be considered permissible to one mathematician may not be so acceptable to another. And as this is compounded by the presence of other mathematical conventions and 'thinking habits' -- there could be a virtual chasm of communication between mathematicians when it comes to considering what is or is not to be considered acceptable, or 'existent and real', in mathematics. And with respect to the presence of these conventions, it would be a minor achievement, at the very least, for a mathematician to read what another mathematician has created and be able to interpret it, indeed even read it, in the same manner as it had been presented.[8] This became a really important consideration

---

[7]Ernst Zermelo, A New Proof of the Possibility of a Well-Ordering, from Van-Heijenhoort, Jean, From Frege to Godel, Harvard University Press, 1967; p.192.

[8]One would have to make a veritable 'leap of faith' in order to go from one mathematical school to another and be able to understand the terminology and reasoning procedures which are employed.

when Cantor introduced his transfinite number theory, because it became apparent that the language with which he spoke did not have a common meaning to the mathematicians who became aware of his theory -- and this was particularly significant with such basic terms to his work as 'finite', 'infinite', and 'set'. That Cantorian set theory utilized ideas and terms that did not have a common meaning is a point which should really be emphasized, for, perhaps due to a lack of clear understanding of it, it seemed that Cantor's creation was often viewed as in a kind of opposition to the (potential) conception of infinity, instead of being seen as a whole new approach.

There is a long-standing convention, and one which is a central problem that Cantor had to deal with (and which shall be developed and expanded upon in a later section), the traditional view of 'mathematics as descriptive'. Herein is concerned the question of reference; namely, is there some mathematical entity which the mathematical theory -- in Cantor's case the theory of transfinite numbers -- refers to? That is -- it was expected that the mathematical theory would have a concrete or abstract referrent which it would pertain to.

Generally, it was the case that mathematics, up until the 1800s, referred to, directly or indirectly, some material entity -- that is, it was some sort of description of the physical world. After mathematics moved on to more abstract levels, or abstract in the sense that it was more abstracted from the physical, it becomes less apparent what would be the conventional bond with the material world. The convention came to be developed that a mathematical theory must still correspond to some 'existent' entity, be it only an abstract one. Thus, when Cantor brought forth his theory of transfinite numbers, a problem which he had to face was: Where or what is the entity or entities which this theory is descriptive of? And it was considerations such as this that led to attempted proofs and later postulates to establish that an actual infinity did 'exist'.

If one returns to the question raised earlier about determining the acceptability of introducing a 'second number system' or any other mathematical creation -- it appears that one comes upon the onion

problem. Namely, if we set aside the conventional viewing of 'infinity' as potential and, instead, take a new direction and approach it from a 'next after the finite' standpoint, then we find ourselves up against even more conventions -- layer after layer of them. And even if one were able to do a kind of phenomenological epoche upon these layers -- that is, even if one were able to 'strip' or 'bracket' off the conventions regarding language, proof procedure, consistency, describability, decidability, and so on -- would it then be possible to determine an answer as to whether or not a new theory is acceptable as a mathematical creation?

In the next section the convention regarding 'mathematics as descriptive' shall be contended with, along with Cantor's attempts to reconcile it with his transfinite number theory.

CHAPTER II

'EXISTENCE AND REALITY'
THE 'MATHEMATICS AS DESCRIPTIVE' CONVENTIONS

> The establishing of the principles of mathematics and the natural sciences is the responsibility of metaphysics. . . The general theory of sets . . . belongs entirely to metaphysics. You can easily convince yourself of this by testing the categories of cardinal number and ordinal type, these fundamental concepts of set theory, with respect to the degree of their generality and also notice that the reasoning about them is quite pure, so that fancy has no room for play.
> This is in no way changed by the pictures which I, like all metaphysicians, sometimes make use of to explain metaphysical concepts. Nor does the fact that my work appears in mathematical journals affect its metaphysical content.[1]

That Cantor came to conclude that his work in set theory was metaphysical, rather than mathematical, seems to be based on reasons in addition to the generality of its content. In particular, it may have been the response among mathematicians to the subject matter -- infinite sets and transfinite cardinal and ordinal numbers -- that led Cantor to question whether his own theory was of mathematical value.

Prior to Cantor's time, it was commonly held that when one spoke of 'infinity', one referred to something which was never quite obtainable. That is, it was viewed as a limit which was approached by the increasing positive integers. Hence, it symbolized the potential -- that the number system is never a completed process, but continues to magnitudes which potentially become, but never reach, the infinitely large.

This convention was generally accepted and unquestioned until Cantor suggested a new one for consideration: What prevents the introduction of a new concept of the infinite as that which is first <u>after</u> the finite? If one could lay aside all conventions in opposition to this 'new way of seeing', one might then be able to determine whether or not it is tenable to contend, with Cantor, that the transfinite

---

[1]G. Cantor, quoted by Meshkowski, H., <u>Ways of Great Mathematicians</u>, Holden-Day, Inc., London, 1964; pp.94-5.

numbers are a 'second number system'.

A central question raised seemed to be one of 'existence and reality'. Many mathematicians, like Kronecker for instance, contended that the mathematical theory is either untenable, or irrelevant, unless there was some existent mathematical entity corresponding to the theory of transfinite numbers and that this existence was constructible, presumably in a finite number of steps. As in the case of imaginary numbers, where Kronecker believed that, "True their use doesn't lead to contradictions. Nevertheless their introduction is unwarranted, for imaginary magnitudes do not exist" --the same charge was aimed against the transfinite numbers. And the attack became intensified as it became apparent that the introduction in this case did lead to contradictions.

The whole conception of transfinite numbers raised suspicions, since it was apparent that one could not reach ω, for instance, by counting. Indeed, even Cantor himself demonstrated this, by showing that there could be no number which immediately preceded ω. He demanded that the old traditional viewing of the infinite as potential be made more flexible, in order that a new convention of infinity be recognized. But what he failed to acknowledge was the strength of the hold of the old conventions.

Cantor suggested that

> . . . mathematics is, in its development, quite free and only subject to the self-evident condition that its conceptions are both free from contradiction in themselves and stand in fixed-relations, arranged by definitions, to previously formed and tested conceptions. In particularly, in the introduction of new numbers, it is only obligatory to give such definitions of them as will afford them such a definiteness, and, under certain circumstances, such a relation to the older numbers, as permits them to be distinguished from one another in given cases. As soon as a number satisfies all these conditions, it can and must be considered as existent and real in mathematics. In this case I see the grounds on which we must regard the rational, irrational and complex numbers as, just as existent as the positive integers.[2]

---

[2]G. Cantor, <u>Introduction to the Founding of a Theory of Transfinite Numbers</u>, trans. by Philip Joudain, Dover Publications, New York, 1915; pp.67-8.

To say on one hand, that mathematics is quite free and, on the other hand, that it is subject to only two conditions, freedom from contradiction and standing in 'fixed-relations' to 'formed and tested conceptions' is to be saying in fact two quite different things. Being quite free could be interpreted to mean that mathematics is free from rigid structure or conventions and quite flexible in determining the acceptability of a mathematical theory. On the other hand, the 'self-evident' conditions could be quite restricting upon the development of any mathematical theory. The initial condition -- namely, freedom from contradiction -- is not only ironic in light of the set-theoretical paradoxes which drew upon the basic concepts of the theory, but also because it is a condition which is not always easy to determine. That is, it could very well be the case that a contradiction is not discovered until long after a theory or proposition, etc. is introduced into mathematics. As a result, it may be rather presumptuous of a mathematician to declare something as 'existent and real' in mathematics. If not that, then he might be forced to withhold judgment indeterminably as he awaits a final word as to whether there may or may not be contradictions in his theory.

Cantor makes a further requirement, or 'self-evident' condition, which must be taken into account: Any new theory must stand in 'fixed-relation' to previously formed and tested conceptions. This could be seen as quite a limitation upon the freedom of development of mathematics, according to what was determined to be the 'fixed-relations' or the 'tested conceptions'. In light of the presence of so many mathematical conventions and the differing demands and conditions within each mathematical school or perspective, one may wonder whether it could be agreed upon at all what would constitute a fixed-relation in mathematics -- much less the controversy which would ensue on determining the 'formed and tested conceptions'. For example, Georg Cantor ran against many ideas of mathematics that were so 'formed and tested' that he found that ". . . people of a timid disposition . . . got the idea that the continuum constituted a rather religious dogma than a logico-mathematical concept".

The problem of deciding upon the fixed-relation condition is not really an easy one. Cantor, taking a rather formalistic stand, seemed to indicate in his statement that the type of 'fixed-relation' that should be sought is one which is based upon (i) definitions and, consequently (ii), the method employed for defining (introducing) any new concepts or theories into mathematics. He seemed to suggest that it is on the ground of definitions that the new conceptions will be understood and distinguished from the 'previously formed and tested' ones. Cantor appeared to mean that mathematics was free insofar as this demand (that they be clearly distinguishable on the basis of definitions) is the only thing to be considered -- that is, it need not be of concern whether it is an expression of any kind of processes or relations in the material world.

Of course the question immediately arises, especially with regard to the mathematical 'thinking habits', as to whether one could actually create a mathematical system that had virtually <u>no</u> 'fixed-relation' to any previous mathematical conception at all; even with respect to the numbers, symbols, and operations utilized. And, furthermore, if one could actually create such a system, could or would it be considered mathematics? It may appear that a necessary condition would be to require an accessible language; a glossary, so to speak, of terms, symbols and so on. Perhaps, even if it were the case that it was completely foreign as a 'language' -- beyond the relativity of the sub-languages of the different mathematical systems -- then it might not cease to be mathematics so much, as it may just cease to be understandable, as far as someone acquainted with traditional mathematics may be concerned. In order that it be considered a form of communication of mathematics, however, it would, on the surface, appear that an accessible 'language' would have to be considered one of the 'fixed-relations' about which Cantor spoke. In that way, the distinguishability on the ground of definitions may essentially be the sort of 'fixed-relation' sought.

The relationship between mathematics and reality, between mathematical theories and 'existing' mathematical entities, seemed

generally to be bound up with a strong convention -- applicability.

> Originally, the function of the theory was to describe certain phenomena in the external world, especially in the three-dimensional space in which we live (or fancy to live; I leave this undecided here). Euclid gave definitions of his fundamental objects . . . where it suffices to give the reader an adequate idea of the concepts, while it is not necessary to mention all its characteristic properties. After having fixed his fundamental concepts by such pseudo-definitions, Euclid introduced other concepts by real mathematical definitions . . . The separations between the ideas of physical and mathematical space was not made, at least not explicitly, before the end of the nineteenth century . . .[4]

That is, before the end of the nineteenth century it was the case that mathematics was almost strictly being viewed as 'mathematics as descriptive' of the physical world. And it was this convention that became the most pressing upon Cantor. Because the correspondence between transfinite number theory and the so-called real (physical) world seems to be vague, if not non-existent, this may have been sufficient evidence for one to conclude that Cantor's work was meaningless as a mathematical work. Furthermore, it was not the case that such questioning ended as mathematics continued to develop in the twentieth century. The question was as present in Cantor's day with Kronecker as it is in our own with Wittgenstein, who suggested that any expression about the infinite is empty "so long as there is no employment for it". In addition, 'such an employment is not: yet to be discovered, but: still to be invented'.[5]

In a more softened Kroneckerian position with regard to the 'existence and reality' question, is Heyting. Heyting contended that an axiomatic foundation of set theory is not conceivable, since we cannot construct a consistent model for the set theory. "If set

[4] A. Heyting, "Axiomatic Method and Intuitionism", from Bar-Hillel, Poznanski, Rabin, and Robinson (eds.), Essays on the Foundations of Mathematics, North-Holland Puc., 1962; p.237.

[5] It is important to realize that Wittgenstein is mainly concerned with the theory's use and not whether or not it is mathematics, mysticism (as Kronecker felt) and so on. In fact, in the Remarks, he asked, "even as a joke, isn't it evidently mathematics"?

theory is not completely void, something must exist that fulfills the axioms, but this existence is only revealed by the axioms". That is, there must exist some sort of mathematical entity, be it concrete or abstract, which the theory describes. But Heyting felt that, "as a mathematical object is only considered to exist after its construction, it cannot be brought into being by a system of axioms".[6] This type of reasoning leads to more subtle and complex problems than merely insuring that a mathematical proposition pertain to some concrete, material object. What is being demanded is a (mathematical) object of reference, which the theory would be referring to.

Although twentieth-century mathematicians generally recognize Cantor's theory as part of mathematics, rather than metaphysical philosophy, many questioned the use of an 'actual' infinite and, like Wittgenstein, wondered what one could _do_ with it. The problem here seems twofold -- on one hand is the upsetted convention of 'mathematics as descriptive' and on the other hand is the upset of an even more specific definition of 'infinity'.

As mentioned earlier, there is a long tradition in mathematics of seeing its subject matter as descriptive -- _i.e._ there is a convention of believing that there must be an employment, in some very real concrete sense, for a mathematical theory after one has gained some understanding of it. Therefore, the problem of transfinite number theory was, what could one do with such information? In other words, what use can be made of transfinite numbers? In mathematics there had been, until the end of the nineteenth century, a prevalent tradition of a somewhat fanatic desire to insure that a correspondence exists between mathematics and reality and consequently an emphasis was placed upon the utility of a mathematical theory. And even later when mathematics developed and expanded along more abstract lines, where many branches were divorced from the physical world and, hence, material application -- the tradition still continued that a theory of mathematics was to be descriptive of some sort of mathematical entity,

---

[6]_Op.cit._, p.239.

whether it be concrete or abstract, materially existent or abstractly existent (postulated existence). That is to say, there had always been the presupposition of a concept which the definitions and axioms of mathematics appeared to construct.

When there failed to be a strong correlation between a mathematical theory and the material world, the customary response at Cantor's time, was to question the legitimacy of the entire mathematical work. This occurred with set theory and it occurred when there was a confusion or discrepancy because of too many conflicting or contrasting theories, as in the case of non-Euclidean geometry. In the case of the non-Euclidean geometers Lobachevski and Riemann, the concern was with investigating possible deviations from Euclidean geometry, with a more or less dispassionate regard for 'reality'. The immediate response to the non-Euclidean theories was based upon a confusion as to what would then constitute the more valid description of 'reality' -- and has proved to be a paradigm case of mathematical tradition.

What is most significant is that mathematics was being seen in a very basic way -- that is, mathematics as descriptive. This does not just signify mathematics as descriptive of the physical world, as in the case of the more applied mathematics. For when 'pure mathematics' developed there was still expected to be a correlation between the mathematical theory and some mathematical entity (with perhaps an only postulated existence). In light of this position, the reception to Cantorian theory is not all that startling, for the discrepancy seemed to be with his theory and the object of its description. Precisely, what was being described by a theory of transfinite numbers? Cantor's set theory was not a mathematical isomorph of any previously existent system of mathematics or reality. Cantor's statement upon 'existence and reality' of a mathematical entity obviously went unheeded and any search to find such an object either led to absurd claims about theology (such as the actual infinity of God's knowledge) or to mathematico-philosophical claims such as Dedekind's actual infinity of ideas and concepts.

In an attempt to establish that infinite sets exist -- and due to 'mathematical proofs establish mathematical truths' conventions Dedekind felt the attempt as necessary -- Dedekind purported to prove that the totality S of all things which can be objects of thought is infinite:

> For if s signifies an element of S, then is the thought s', that s can be an object of my thought, is itself an element of S. If we regard this as Ø of S, thus determined, the property, that the transform S' is a part (subset) of S; and S' is certainly a proper part of S, because there are elements in S (e.g. my own ego) which are different from such thoughts s' and therefore are not contained in S'. Finally it is clear that if a, b are different elements of S, their transforms a', b' are also different, that, therefore, the transformation Ø is a distinct (similar) transformation. Hence S is infinite . . . [7]

And then later there were 'proofs' -- such as Russell's statement that '. . . to say, for example, that a certain length of time elapses between sunrise and sunset is to admit an infinite whole, or at least a whole which is not finite -- which attempt to demonstrate that there must, in some concrete or abstract way, exist infinite amounts. Why such 'proofs' were even attempted, or deemed necessary to attempt, is a matter of convention. Furthermore, it seems that one of the goals of such attempts is to provide an entity which would correspond to transfinite number theory -- and as a consequence the theory could be viewed as 'descriptive'. Difficulties (e.g. circularity of argument, lack of precision, etc.) with which so-called proofs helped lead to the resulting general acceptance of the Zermelo-Fraenkel Axiom of Infinity, which postulates an infinite set, and, in that way, provides the long sought-after object of description -- be it an unconventional method of achieving (via discovery or invention) a mathematical entity to give 'substance' to Cantorian theory.[8]

---

[7]Richard Dedekind, Essays on the Theory of Numbers. Open Court Publishers, Illinois, 1948; p.64.

[8]The Zermelo-Fraenkel Axiom has been stated by Fraenkel (p.32 Abstract Set Theory) as: 'There exist infinite (reflexive) sets'.

To the more skeptical eye, the postulatory method may very well be compounding, rather than alleviating, the situation. Tradition had already been upset by presenting a mathematical theory and _then_ seeking an entity for it to describe. But it is even more extreme to resort to postulating such an entity's existence, for that is completely overturning mathematical legitimacy -- and it may also lead one to wonder if such an axiom might be merely a pacificatory move.

To postulate existence, like a simulated 'action', is to provide a somewhat artificial entity. Furthermore, it seems to be retreating to the 'mathematics as descriptive' mold -- insofar as it is creating, via a postulated existence, a kind of subject matter for the theory to be describing -- and, henceforth, give it a kind of actuality, and post hoc conventionality, it had not hitherto possessed.

Why is it desirable, indeed necessary, to assume -- by postulates or proofs -- that an infinite set exists? In the times when mathematics was more physically oriented -- _i.e._ when it was directly applicable to the external world -- the emphasis was consequently put on the usefulness, in a very concrete sense, of the theories of mathematics. As it developed into more abstract areas (hence labeled 'pure mathematics') applicability decreased in importance and in some respects ceased to be a consideration at all. Along with that, the concern for having an entity which the theory actually describes seems to be one which should also be expected to decrease in significance. In fact, however, a convention has been developed that retains this descriptive character -- for when concretely existing entities could not be found to correspond, the response was that of mathematically postulating the entities' or entity's existence. It seems such a superfluous move, for the only change it introduces into the mathematical theory is to provide a mathematical referrent, such as the actual infinite sets, which one can now _assume_ to have a kind of existence. If it does not matter whether or not the 'actual infinite' exists, why go through the hoorah of postulating existence?

Perhaps by postulation the theory is made more palatable to the

tradition-bound mathematician. But, it seems that any attempts made to press description onto or into the theory, whether it be by proofs or axioms, is a type of recognition of the Kroneckerian position which epitomizes conventional response to any mathematical creation. It seems that the concept of mathematics as not descriptive is rather repugnant; for the presupposition of an 'existent' entity corresponding to the theory appears to be completely necessary in order that the theory have any meaning at all. That is, without a referrent having some kind of assumed existence, the theoretical stance taken by some mathematical innovator, such as Cantor, would be designated as a rather obtuse game.

Cantor himself seemed to have gotten involved with the 'mathematics as descriptive' convention with respect to the whole question of existence of the 'actual' infinite. This mathematical convention and, inclusive within it, the mathematico-philosophical tradition of Platonism, obviously had its toll on Cantor, as the chapter on his set theory will further demonstrate.

Indeed, Cantor offered a justification for the introduction of his new conception of the infinite and, in that way at least, bowed to the convention of establishing the 'existence and reality' of the mathematical entity -- in Cantor's case, the transfinite numbers:

> We may regard the whole transfinite numbers as 'actual' in so far as they, on the ground of definitions, take a perfectly determined place in our understanding, are clearly distinguished from all other constituents of our thought, stand in definite relations to them, and thus modify, in a definite way, the substance of our mind . . .[9]

When Cantor made this statement (1882) he was a supporter of a formalistic number theory which held that the process of concept formation is one which begins with a sign or symbol firstly and given 'properties' (predicates) which will then determine the relation to other conceptions.[10] When Cantor spoke about the 'existence and reality' of a mathematical creation, he believed that the introduction of a con-

---

[9]G. Cantor, Introduction to Theory of Transfinite Numbers, Dover Publications, 1915; p.67.

[10]Ibid., p.69.

cept (or more precisely, what was to be a concept) followed this formalistic process. Cantor's later work (e.g. his work on set theory in 1895) was along more psychological lines -- and he was quite critical of the formalistic approach to number theory which vonHelmholtz and Kronecker exhibited in their work on number concepts). Considering the remarks Cantor made upon the justification of introducing new concepts into mathematics, whether he tended to any one specific perspective of mathematics (such as formalism), he was obviously affected by mathematical tradition -- for not only did he offer an 'apology' for his creation, but he followed convention in attempting to justify his introduction by trying to demonstrate the acceptability ('existence and reality') of the transfinite numbers.

In addition to the conventions regarding the actuality of transfinite numbers or the 'actual' infinite as mathematical entities, Cantor had to face at least as great a 'thinking habit' or convention with respect to 'infinity' itself. That is, Cantor had to deal with traditional views on how 'infinity' should be defined -- for Cantor was taking a new approach on the subject and was faced with opposition to his perspective. It had long been the case that 'infinity' was never seen as anything actual, but only as a potential growing magnitude, a 'figure of speech' as Gauss contended. There had been developed and perpetuated such a habitual mode of thinking on such a topic, that to even tolerate a new approach was almost unheard of.

In the next chapter the concept of 'infinity' shall be investigated along with the rigidity of conventions surrounding it. Perhaps one can then understand that Cantor's approach, rather than being in conflict with traditional views on the subject offered a new dimension, so to speak, on the issue of the infinite.

CHAPTER III

INFINITY -- POTENTIAL OR ACTUAL ?

No one shall expel us from the paradise Cantor has created for us.[1]

> Imagine set theory's having been invented by a satirist as a kind of parody on mathematics. -- Later a reasonable meaning was seen in it and it was incorporated into mathematics. (For if one person can see it as a paradise of mathematicians, why should not another see it as a joke?)[2]

And to the question: 'Joke or not, is it not mathematics?' there are many mathematicians, such as Kronecker, Poincaré and Gauss, who would answer 'no'. It was, for example, Kronecker's contention that any assumption of an actual infinite and, hence, that infinite sets exist in mathematics, was untenable. He argued against the 'clouding' of mathematics by any such assumption, claiming that the natural numbers and any operations with them are 'intuitively founded'. Furthermore, Kronecker suggested that Cantor's theory of the transfinite numbers was mysticism -- a cry which was similarly echoed by Poincaré at a later date, when he referred to it as a 'disease from which one has recovered'.

It would also appear that one could align Gauss with Kronecker and Poincaré on the issue of the 'actual' infinite. Gauss protested against using infinity 'as if it were something finished' and in 1891 he claimed ". . . this use is not admissible in mathematics. The infinite is only a façon de parler. . . ."[3] Gauss' statement, however, is suggestive of the Aristotelian view that mathematicians have no need of the infinite and do not make use of it, but only ask (postulate) that the finite become as long as they wish, as something endless or indefinitely great.

---

[1]David Hilbert, as quoted by Kline, Morris, Mathematics in Western Culture, Oxford University Press, N.Y., 1953; p.397.

[2]L. Wittgenstein, Remarks on the Foundations of Mathematics, p.137.

[3]Gauss, as quoted by Fraenkel, Abstract Set Theory, North-Holland Publishers, 1961; p.1.

This is the kind of mathematical 'thinking habit' whereby theories and mathematical entities are thought of and evaluate in terms of 'needs', 'uses', and objects of description, that lead to view the infinite as a mere figure of speech and only a 'potential'. It was in opposition to this kind of position that Cantor came to see a need for the infinite, as far as the development of his theory necessitated. As a matter of fact, he came to believe that such a use was absolutely necessary, for he stated:

> . . . my investigations in the theory of manifolds has arrived at a spot where the continuation becomes dependent upon a generalization of the concept of real integer beyond the usual limits; a generalization which takes a direction that, as far as I know, nobody has yet looked for.
> . . . the venture is to generalize . . . the series of real integers beyond the finite. Daring as this might appear, I do express not only the hope but the firm conviction that in due time this generalization will be received as a quite simple, suitable, and natural step. Still I am well aware that by taking this step I put myself in a certain opposition to widespread views of the infinite in mathematics and to current opinions regarding the nature of number.[4]

That Cantor should feel so hesitant, that Cantor should feel so apologetic in the development of his theory of transfinite numbers, indicates his awareness of mathematical conventions. The statements Cantor made in this respect, further indicate that the strength of these conventions create a type of inhibitory reasoning -- that is, there seems to be a sort of uneasiness regarding the introduction of new mathematical creations. In the same way that Wittgenstein noted the 'deep need for convention' in mathematics and Gauss demonstrated in his refusal to publish his work on non-Euclidean geometry, Cantor recognized the power of tradition in mathematics. However, Cantor's inventions, or discoveries, convinced him of their importance and the necessity of proceeding. Knowing that he was going against some firm conventions, Cantor claimed

> I was logically forced, almost against my will, because in opposition to traditions which had become valued by me in the course of scientific researches extending over many

---

[4]Ibid., p.3.

> years, to the thought of considering the infinitely great, not merely in the form of unlimitedly increasing, and in the form, closely connected with this, of convergent infinite series, but also to fix it mathematically by numbers in the definite form of a 'completed' infinite. I do not believe, then, that any reasons can be urged against it which I am unable to combat.[5]

It was as a consequence of this 'new way of seeing' into the number system, that Cantor introduced a new direction and a new conception -- the transfinite numbers. And even in his early work of 1882 one sees Cantor's attempt to 'fix' the infinite mathematically. Furthermore, one finds such a usage (i.e. of the completed infinite) in his conception of the 'definitely defined symbols of infinity'.

What Cantor suggested, and which had apparently hitherto been ignored by mathematicians, was that the infinite numbers must be viewed in an entirely different light from that of the finite numbers. Indeed, the problems which had previously been encountered with respect to transfinite numbers had arisen because such theorems and proofs began 'by attributing to the numbers in question all the properties of finite numbers, whereas the infinite numbers, if they are to be thinkable in any form, must constitute quite a new kind of number as opposed to the finite numbers'.[6]

That is, one must leave behind the old mathematical 'thinking habits' and depart from the traditional way of ascribing parallel characteristics of the transfinites to those of the finites, in order to conceive of the transfinite numbers at all. It is only then that one can arrive at different systems which would be both logically possible and fruitful in dealing with the non-finite numbers. Although one cannot assert a greatest finite number, one can set aside tradition and postulate a new transfinite number -- that which is "first after the finite numbers". By such a procedure, Cantor was able to

---

[5]G. Cantor, Contribution to the Founding of a Theory of Transfinite Numbers, Dover Publications, 1915, N.Y.; p.53.

[6]G. Cantor, Contribution to the Founding of a Theory of Transfinite Numbers, Dover Publications, 1915, N.Y.; p.74.

avoid many of the problems and contradictions which earlier mathematicians had run into, when they failed to make a distinction between the finite and the transfinite numbers.

There were a number of ways in which mathematicians did not succeed in making a clear distinction between the finite and the infinite. It seemed as if the infinite was perpetually being viewed as some strange perversion of the finite. Take, for instance, the view of Poincaré with respect to what he considered the two conventional opinions of infinity:

> For some, infinity is derived from the finite, infinity exists because there is an infinity of possible finite things. For others, infinity exists before the finite; the finite is obtained by cutting out a small piece from infinity.[7]

That is, Poincaré conceived of the infinite as interlocked in some way with the finite, insofar as one concept was the derivation of the other. Further, Poincaré's approach is such that it seems to presuppose some notion of the infinite before one can understand the two 'opinions' in the first place. And although the concept of infinity is obviously related to that of the finite, it seems that the understanding of the infinite need not be so connected with the finite. Namely, why can't the infinite be seen separately, as a new consideration, instead of a 'variable finite', as Cantor put it?

Cantor contended that the derivation approach of which Poincaré spoke, had created many of the problems with respect to understanding the concept of infinity.

In order to insure that infinite numbers were conceived of as having different characteristics from the finite numbers, Cantor introduced the symbol 'ω' as a replacement for the symbol '∞'. Not only did Cantor want to create a new convention of 'thinking' about infinity, but he wanted to clarify, via a new kind of language, that this conception was, in essence, a 'new way of seeing'. There were so many preconceptions/opinions surrounding '∞', that a new symboliza-

---

[7] Henri Poincaré, Mathematics and Science: Last Essays, Dover Publications, N.Y., 1963; p.65-66.

tion might help in dispelling some of them, initially at least. In addition, if Cantor wanted to create a 'new way of seeing', then he must also create a new way of symbolizing and talking about this new entity, in order for the old thinking habits to be relaxed, so that a new kind of understanding could take place.

It was Cantor's contention that '∞' represented an 'improper' infinite, whereas 'ω' represents what Cantor called the 'proper' infinite. This notion of an improper and a proper infinite was brought forth in 1883 in the Grundlagen einer allgemeinen Mannichfaltigkeitslehre, where Cantor suggested that these are the two ways of conceiving of infinity. He labelled as the 'improper' infinite that which increases without limit or decreases arbitrarily small -- but in either case remains finite. This is the way Poincaré and mathematicians prior to Cantor spoke of the infinite -- colloquially it may be stated "infinity is like the finite -- only more of it". Cantor conceived of the infinite in another sense, the 'proper infinite'. 'Proper infinite' is represented by the point infinity of the complex plane (in the theory of functions). In this case, one would study the behavior of functions about this definite point, in the same manner as about any other point, since

> The behavior of the function in the neighborhood of the infinitely distant point shows exactly the same occurrances as in that of any other point lying in finito, so that hence it is completely justified to think of the infinite, in this case, as situated in a point.[8]

The conception of the 'proper' infinite was the one which Cantor centered his ideas of the transfinite numbers upon. And the usage of 'ω' instead of '∞' was to render the clarification more complete, so with the new symbol some of the old preconceptions could possibly be eliminated.

Cantor suggested that, if one were so inclined as to make a comparison at all with the transfinite numbers, it should be the ir-

---

[8] G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.56.

rational numbers which should be examined. Cantor believed that a strong relationship existed between the transfinite numbers and the finite irrationals, because, 'in their inmost being they are all alike, for both are definitely marked off modifications of the actually infinite'. Indeed, earlier in his article of 1884, Cantor had said

> ω is the largest transfinite ordinal number which is greater than all finite numbers; exactly in the same way that $\sqrt{2}$ is the limit of certain variable, increasing, rational numbers . . . . The transfinite numbers are in a sense new irrationalities, and indeed in my eyes the best method of defining irrational numbers is the same in principle as my method of introducing transfinite numbers.[9]

To make such a comparison and to imply that transfinite numbers are 'new irrationalities' adds quite a new aspect, a new way of 'thinking', on the matter. When Poincaré spoke of the ways of looking at the finite/infinite, at no point did he mention the irrational numbers at all, a consideration which might prove noteworthy, since both transfinite numbers and irrational numbers stand apart in many ways from the finite rationals. Both are distinct in the manner in which they can be spoken of and dealt with.

However, the matter is not all that elementary. The irrational numbers of the form $\sqrt{b}$, where b is a positive non-square integer, have one very significant attribute. Namely, if one were to square such an irrational, one would again encounter a mere finite rational. And this would produce the 'older numbers' (finite rationals), it would seem that Cantor's comparison may be enlightening somewhat but the analogy fails under certain considerations. Furthermore, where we can speak of $\sqrt{2}$ as a limit of certain variable, increasing rational numbers, we cannot do so with ω, i.e. we must use a different kind of terminology altogether. For although ω may be seen as the least of all numbers which are greater than all the finite numbers ν, it is not the case that lim $(\omega - \nu) \longrightarrow 0$ as ν increases. Indeed, $\omega - \nu$ is always equal to ω, and 'any number ν however great is quite as far off

[9] G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.77

from ω as the least finite number'. And consequently, 'ω is not a maximum of the finite numbers, for there is no such thing'.[10]

There are, as one can see, a number of dissimilarities between the transfinite and the finite irrational numbers.[11] And any attempt to draw a parallel between the ways in which the two are viewed, is bound to disintegrate. It should be kept in mind, however, that they are alike in several respects.[12] Not only are there a number of complexities with respect to conceiving of either of the two mathematical entities, but both are noticeably distinct from the finite rational numbers. This was shown too because of the combination of both fascination and disdain that mathematicians heaped upon both creations as

---

[10] G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.78.

[11] With regard to dissimilarities between finite irrationals and transfinite numbers some examples are:

1. Trivially, the number(s) is (are) finite while the other(s) is (are) infinite.
2. Finite irrationals of the form, $\sqrt{b}$ where b is a non-square positive integer, can be operated upon (i.e. squared) to obtain a finite rational, whereas this can never be the case for the transfinite numbers.
3. Besides the above (No.2), there are differences with respect to the arithmetic operations, which would not hold with the respective cases for the finite irrationals --

   e.g. $\omega - v = \omega$ (v finite ∞ $v > 0$)

   $\omega / v = \omega$ (v finite ∞ $v > 0$)

   $\omega \times v = \omega$ (v finite ∞ $v > 0$) etc.

   none of these would hold if a finite irrational were substituted for ω in the above equations.
4. Finite irrationals have a geometric interpretation on the real number line.

[12] What is particularly being meant in the case of the similarities, is in regard to the response which took place in mathematics -- i.e. both creations caused alarm/disdain upon their introduction into mathematics. For instance, in the case of the finite irrational numbers, one might look at Pythagorus and his mathematical school of Crotona, wherein irrational numbers were discovered (created) and the cause of much disturbance. There were efforts made to keep their discovery (creation) secret, since it not only went against intuition and/or common sense, but showed that they resulted in incommensurable line segments.

they were introduced into the mathematical sphere. For, in both cases, the introduction was in opposition to the traditional thinking regarding the nature of number.

Although the challenge to mathematical 'thinking' may have been the same in the case of the irrational numbers and the transfinite numbers, the latter posed a greater problem in light of mathematical preconceptions in this sense: Infinity had been spoken of, utilized, and/or considered by mathematicians for centuries in a rigid mathematical sense; as a potential, a symbol, a 'figure of speech'. And it was in the presence of this long-standing convention that Cantor wanted a new one to be considered. Too, it was not so much that the old convention was to be overthrown, but that a separate convention, a 'new way of seeing', be permitted to exist concurrently. But even to allow a new convention regarding the infinite to be introduced proved to be a matter of much difficulty and confusion. Cantor tried to clarify that he was not trying to replace the view of the infinite as potential, but he found that

> . . . in spite of the essential difference between the concepts of the potential and the actual infinite, the former meaning a variable finite magnitude increasing beyond all finite limits (like x in 1/x), while the latter is a fixed constant magnitude lying beyond all finite magnitudes, it happens only too often that they are confused.[13]

In order to better understand the distinction between the potential and actual infinite and how any confusion could arise between these, one m ust have some understanding of some of the basic concepts of the set theory. In the next chapter, these shall be examined, for only then can one gain some knowledge of what Cantor's 'new way of seeing' entailed and where difficulties (e.g. the basis for the antinomies) arose. What should be kept in mind in the following discussion, is the influence, on the one hand, of some important mathematical and philosophical ways of thought while, on the other hand, how Cantor utilized some of those conventions to an extent never previously done.

---

[13]G. Cantor, quoted by E.T. Bell, Men of Mathematics, Simon & Schuster, New York, 1937; p.556.

CHAPTER IV

BASIC SET THEORETICAL CONCEPTS MATHEMATICAL AND PHILOSOPHICAL

Since Cantor's work on transfinite numbers is intricately combined and concerned with his set theory -- for both transfinite cardinals and transfinite ordinals -- to grasp one we must have an understanding of the other. For, as Cantor indicated, his work into set theory necessitated constructing a theory of transfinite numbers and, therefore, going 'beyond the usual limits'.

What, precisely, was it in his set theory that led Cantor to the precipice between the 'potential' and the 'actual'? First of all, one should consider Cantor's (Platonistic) definition of 'set'. A 'set' or Menge was defined as 'any collection into a whole M of definite and separate objects m of our intuition or our thought. These objects are called the 'elements' of m'.[1] This definition of set indicates that anything can constitute an element of a set, since anything can be the object of our intuition or our thought. More basic to the idea perhaps, is that 'element' is to be the primitive concept; the atom of the molecule, so to speak. By Cantor's definition one cannot speak of the set in any simpler terms. The definition is further laid out as basic by the restriction that the 'elements' are definite and separate. With this limitation, one can avoid the type of infinite sets such as (1, 1, 1, 1, 1, , , . . .) -- <u>i.e</u>. ones in which one distinct element is repeated indefinitely.

There is a certain generality and lack of precision about Cantor's definition of 'set'. Consequently, mathematicians have never been very content with his presentation and, hence, have attempted to retain the fundamentality while at the same time attempting to instill more clarity and 'polish' to it. It seemed that there could surely be found a preferable approach to introducing such a basic concept to the theory. It became apparent that there was a conflict between the

---

[1]G. Cantor, <u>Introduction to Founding of the Theory of Transfinite Numbers</u>, Dover Publications, 1915; p.85.

rigour that was sought and that which could be achieved in clarifying the conception. It was the case that

> . . . a definition in the proper sense means an explanation of a notion by means of more primitive or previously defined notions. However, it is evident that the notion 'set' is too fundamental for such an explanation.[2]

Thus, although mathematicians may have found Cantor's definition of a 'set' to be unsatisfactory, most attempts to render it more palatable have not succeeded very well.

It reached the point that such mathematico-logicians as Fraenkel deemed it 'inevitable' to renounce any such definition at all. Fraenkel suggested three 'remedies': (1) to conceive of the concept of set in a very narrow sense (and consequently much of analysis, geometry, set theory would become meaningless) or (2) adopt a reform of logic as the basis of mathematics, or (3) (the only 'real' choice) take recourse to the axiomatic method. This is a method which Zermelo, for one, advanced.

Zermelo rejected Cantor's manner of defining the set, suggesting an alternative: One must first consider (postulate) a domain $\mathfrak{B}$ of any type of objects, whereby

> Certain fundamental relations of the form $a \in b$ obtain between the objects of the domain $\mathfrak{B}$. If for two objects, a and b the relation $a \in b$ holds, we say 'a is an element of the set b', 'b contains a as an element', or 'b possesses the element a'. An object b may be called a set if and -- with a single exception -- only if it contains another object, a, as an element.[3]

There are a number of differences that should be noted about Zermelo's method of defining a set, as contrasted with Cantor's. Further, the 'existence' convention -- i.e. 'mathematics as description' -- is reappearing. Most obviously, is the change that, in Zermelo's set theory, sets are not simply collections, but are objects which satisfy certain axiomatic conditions. Zermelo believed that it

---

[2]Thoralf Skolem, Abstract Set Theory, Notre Dame Mathematical Lectures, #8, Notre Dame, 1962; p.2.

[3]Ernst Zermelo, Investigations in the Foundations of Set Theory I, from Van-Heijenhoort, Jean, From Frege to Godel, Harvard University Press, 1967; p.201.

was necessary to impose certain restrictions upon the Cantorian notion of 'set'. In Zermelo's theory, and Fraenkel's later, a certain domain of entities, of objects is postulated for consideration. If a certain primitive relationship $\in$, which is referred to as the membership relation, exists between any two of the objects postulated -- say, for example, $x \in y$ -- then x is said to be an element or member of y. However, as Poincaré pointed out, if one did not have an idea of a set to begin with, one would not know anymore by learning that it can be defined in terms of symbol $\in$ (which is also taken as primitive or undefined in Zermelo's definition). And without a prior conception of what a set (Menge) is, the symbol $\in$ would not be at all self-evident. Indeed, $\in$ would be devoid of meaning. Thus, some kind of 'intuition' into the idea of set (Menge) is needed. "But what can this intuition be if it is not Cantor's definition which we have scornfully rejected"?[4]

The questions raised by Poincaré undercut any distinction between the formal approach and the psychological approach to definition in mathematics. On the one hand is the formal definition such as Zermelo's, which tends to be rigorous and mathematically precise (although there have been difficulties in even this method of defining 'set'). On the other hand, there is an approach to defining which is directed more along the lines of 'intuition', <u>i.e</u>. the psychological approach. This is the perspective Cantor took throughout his work on transfinite number theory -- as can be seen in all of his definitions (such as that of 'set', 'cardinal number', 'ordinal type', etc.). It is very general and has been objected to as being imprecise -- for example, that his defining of the basic concept 'set' was not a restricting or a limiting one. Furthermore, it suggests no clear lines of demarcation as to what could be brought into consideration under this concept. It could be argued, however, that to require the kind of clarity and precision needed so that one could utilize it as a rigorous

---

[4]Henri Poincaré, <u>Mathematics and Science: Last Essays</u>, Dover Publications, N.Y., 1963; p.56.

guideline, would be to place upon it demands which perhaps only a more formal definition could satisfy.

A number of mathematicians did object to Cantor's definition of the primary notion of his theory -- i.e. the concept of 'set' -- and pointed out some of the inadequacies. One complaint was raised, for instance, by Poincaré, that the definition of 'set' is a non-predicative one. That is,

> . . . one makes use of the so-called objects in such a way that the totality of these objects, or objects logically dependent upon that totality, are considered as belonging to the same totality so that the definition has a circular character. It might be better to say that a non- predicative definition is the definition of an entity by a logical expression containing a bound variable such that the defined entity is one of the possible values of this variable.[5]

According to mathematical 'thinking' the lack of rigidity and restrictions in the Cantorian definition of 'set' became catastrophic: There were found to be contradictions. That is, there were contradictions which arose from the principle concept of the theory -- the definition of the term 'set'. But other basic terms were also involved.

After introducing the concept of a set, Cantor soon reached the point where 'a generalization of the concept of real integer beyond the usual limits' became necessary. He found that one of the most significant attributes of any set is that of its 'power' or 'cardinal number'. This Cantor had defined as, ". . . the general concept which, by means of our active faculty of thought, arises from the aggregate M when we make abstraction of the nature of its various elements m and of the order in which they are given".[6] What this 'double act of abstraction' involves is utilizing the mathematical 'thinking habit' of the ability to abstract and group objects together. Furthermore, it requires that one be able to abstract twice -- namely,

---

[5]Thoralf Skolem, Abstract Set Theory, Notre Dame Mathematical Lectures, #8, Notre Dame, 1962; p.52.

[6]G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications,N.Y., 1915; p.86.

one must be able to abstract from any specific characteristics of the individual elements of the set. Then one must abstract from (ignore) the order of the elements -- so that what is first, second or last, is irrelevant for the understanding of the cardinality of the set. If one is thus able to abstract from the kinds of elements (whether they are numbers, etc.) and the order in which they are presented, then one is left with the 'cardinal number' of the set. This method of defining cardinality is not only imprecise mathematically since it is more of a descriptive rule, but it is also contingent upon certain aspects of mathematical 'thinking' (e.g. the Platonistic view toward the nature of number) which are presupposed for the definition to be understandable. It should be noted, however, that this approach, again like that of defining 'set', is a psychological one and hence without the rigor which later, formal definitions, entailed. As Jourdain commented:

> . . . Cantor defined 'cardinal number' and 'ordinal type' as general concepts which arise by means of our mental activity, that is to say, as psychological entities. . .[7]

The concept of cardinality has been formally defined and better clarified by the notion of equivalence. Two sets are considered 'equivalent' if there exists a one-to-one correspondence between their respective elements. That is, there must be some sort of function or relation in which to each element of one set there corresponds exactly one element of the other. It is this equivalence of sets which forms the necessary and sufficient condition for the equality of their cardinal numbers.

In order to explain the concept of (finite) cardinal number, Cantor sets forth in the following manner: If there is a set E_o which contains exactly one element e_o (thus $E_o = (e_o)$) there will correspond as cardinal number what we call 'one', hence $1 = \overline{\overline{E}}_o$, where $\overline{\overline{E}}_o$ denotes the cardinal of E_o. We will form E_1 by taking the union of E_o with another thing e_1, so that $E_1 = (E_o, e_1) = (e_o, e_1)$. The cardinal of

---

[7]G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.202.

E_1 is called 'two'. Similarly, all the other E_i and $\overline{\overline{E}}_i$ are obtained. And, consequently, $\overline{\overline{E}}_v = \overline{\overline{E}}_{v-1} + 1$.[8]

What should be recognized in this approach is that the whole explanation could be seen as meaningless, incorrect, or correct according to which convention is used in how it is to be 'read' and what the symbols can be seen to represent.

Utilizing the concept of cardinality [denoted as $\overline{\overline{M}}$ for the set M] in order to categorize sets in terms of 'finite' and 'infinite', Cantor proceeded as follows: "Aggregates [sets] with finite cardinal numbers are called 'finite aggregates [sets]' all others we will call 'transfinite aggregates [sets]' and their cardinal numbers 'transfinite cardinal numbers'".[9] Cantor's statement is suggestive of another long-standing mathematical convention, the 'negative' approach to defining infinite sets. Traditionally, mathematicians defined and understood the concept of infinite numbers and infinite sets as a consequence of what the finite is not. This had been a perfectly acceptable mathematical convention with respect to the introduction of the infinite. A more direct approach and a new convention in this regard was created by Bolzano and Dedekind.

> A set R is called infinite and more strictly reflexive, if R has a proper subset that is equivalent to R. A set which is not reflexive is called finite . . .[10]

This was presented as a theorem by Cantor and used as the definition of 'infinite set' by Dedekind and Pierce. Furthermore, this property of infinite sets was termed the 'paradox of the infinite' by

---

[8]It should be recognized that Cantor's explanation of cardinal number contains some confusion with respect to notation -- for he utilizes the same notation -- parentheses -- to mean both union and set notation. Further, such confusion of set and union notation, results in confusion of the notions of 'element' and 'set'. This, of course, further emphasizes a point made earlier about the 'seeing' of proofs or statements -- i.e. how the conventional notation, etc. is 'read' is very important for deeming the explanation meaningful, correct, incorrect, etc.

[9]G. Cantor, Contributions to the Founding of a Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.104.

[10]A. Fraenkel, Abstract Set Theory, North-Holland Pub., Amsterdam, 1958; p.29.

Bolzano -- although it can only be deemed 'paradoxical' in light of the ordinary ways of thinking, such as that 'the whole is greater than any of its parts'. As a matter of fact, because of such 'paradoxes' as one-to-one correspondence between the set of positive integers and the set of even integers, which Galileo first noted, mathematicians felt that any operations with infinite sets were bound to be questioned. As Russell noted with regard to Galileo's discovery

> This property was used by Leibniz (and many others) as a proof that infinite numbers are suspicious; it was thought self-contradictory that "the part should be equal to the whole". . . Those who have regarded this as impossible have, unconsciously as a rule, attributed to numbers in general properties which can only be proved by mathematical induction, and which only their familiarity makes us regard, mistakenly, as true beyond the region of the finite.[11]

That certain attributes of infinite sets should be deemed 'paradoxical' and suspicious, arose generally as a consequence of the mathematical conventions whereby the infinite was being thought about and investigated in the same manner as the finite.

One of the results of the property, or definition, of the infinite (that of Dedekind's above), is that it gave a new direction to mathematical 'thinking' about the infinite, for it gives a kind of priority to the infinite set, in the sense that -- contrary to the approach of Cantor and most other mathematicians -- it defines a finite set in terms of what the infinite set is not.

This is indicative of a whole trend which had been present throughout mathematical history. Whenever the infinite was spoken of, it was presented as a kind of negation or antithesis of the finite; in the same way in which Poincaré spoke of the derivation of the infinite from the finite. Analogously, the divergence of sequences and series has usually been introduced and referred to as that which does not converge, that which results when the term 'convergent' is not applicable, or fails, for a series. This same type of 'thinking'

---

[11]Bertrand Russell, Introduction to Mathematical Philosophy, Geo. Allen and Unwin Ltd., New York, 1919; pp.80-81.

occurred with respect to the infinite, for it had generally been viewed as a kind of failure of the finite.

This type of attitude or predilection in mathematics in many ways proved to be a kind of deterrent as to the actual investigation of such a concept as the infinite. That is, it had usually been in the shadow of the finite, hence of little significance as a separate consideration. However, once the concept of infinity was investigated for its own sake, as Cantor and later mathematicians discovered, examples of infinite sets and infinite numbers were soon to be found. For example,

> We meet the true infinite when we regard the totality of numbers 1, 2, 3 . . . itself as a completed unity or when we regard the points of an interval as a totality of things which exist all at once. This kind of infinity is known as actual infinity.[12]

The set which Hilbert above refers to as an example of an 'actual' infinite set, is the collection of all finite cardinal numbers v. The cardinal number of this set $\{1, 2, 3, \ldots\}$ is called, after Cantor, 'aleph-null', and is denoted as $\overline{\overline{\{v\}}} = \aleph_0$. In order to demonstrate that $\aleph_0$ is transfinite, Cantor utilizes the Dedekind approach, in noting that 'if to the aggregate $\{v\}$ is added a new element, e_0, the union-aggregate $(\{v\}, e_0)$ is equivalent to this original aggregate $\{v\}$'.[13] Because of the property that the cardinal of two (disjoint) sets is the sum of the respective cardinals, it results that $\aleph_0 + 1 = \aleph_0$, and since $\alpha + 1 \neq \alpha$ for any finite number α, it follows that $\aleph_0$ is not equal to any finite number and hence infinite.

When given a set, one may not only be able to determine the cardinality, but also one may be able to discern a way (or ways) in which the elements of the set can be placed into a kind of order. For instance, there is the set of letters ordered alphabetically, or a set of people ordered by age or weight. There are many ways one

---

[12] David Hilbert, On the Infinite, from Benacerraf and Putman (eds.), Philosophy of Mathematics, Prentice Hall Inc., New Jersey, 1964.

[13] G. Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.104.

could order a set of numbers, for example:

(1) $\{1,2,3,\ldots\}$ ordered according to magnitude

(2) $\{\ldots,3,2,1\}$ ordered according to decreasing magnitude.

Cantor has defined simple ordering in the following manner:

> We call an aggregate M 'simply ordered' if a definite order of precedence rules over its elements m, so that, of every two elements m_1 and m_2, one takes the 'lower' and the other the 'higher' rank, and, so that, if of three elements m_1, m_2, m_3 say, m_1 is of lower rank than m_2, and m_2 is of lower rank than m_3, then m_1 is of lower rank than m_3.[14]

In other words, 'order' can be seen as a relation of precedence, on the elements of a set and one which obeys the laws of transitivity and trichotomy. It should be noted that the order relation, which is denoted as $\prec$, is not the same as the arithmetic relation of 'less than' -- for, whereas 'less than' is an ordering relation and one often found, for instance, with the natural numbers -- the ordering relation $\prec$ is a prescription, or operation, of far more generality. It can designate any one of a number of ways whereby a set could be ordered.

In conjunction with the concept of the order of a set, is that of the ordinal type, which is denoted as $\overline{M}$ for the set M. "By this we understand the general concept which results from M, if we only abstract from the nature of the elements m, and retain the order of precedence among them".[15] What this means, is that it is not the elements themselves that we are concerned with, so much as the way in which a set has been ordered. Earlier, when speaking of the cardinality of the set, it was observed that Cantor regarded it as a 'double act of abstraction'. Cantor had suggested that by abstracting from the ordinal type the order of precedence of the elements, one is left with the cardinal number of the set. With cardinality, one is neither concerned with the type of elements themselves (whether they be integers, rationals, oranges, or what-have-you)

---

[14] _Ibid._, p.110.

[15] _Ibid._, p.112.

nor with the ordering relation between the elements. So with the concept of ordinal type, we would only consider a 'single act of abstraction'. Symbolically, this can be stated: Given a set M, if we execute one act of abstraction we obtain $\bar{M}$, the ordinal type, and if we execute a double act of abstraction, we obtain $\bar{\bar{M}}$, the cardinal number of the set M.

As with the case of cardinality and other concepts Cantor sought to define, ordinal type is not given a very formal definition. Later mathematicians attempted to give the concept a more formal structure. For example, von Neumann tried to give "concrete form" to Cantor's notion of ordinal number:

> Ordinarily, following Cantor's procedure, we obtain this notion by "abstracting" a common property from certain classes of sets. We wish to replace this somewhat vague procedure by one that rests upon unequivocal set operations. The procedure will be presented . . . in the language of naïve set theory, but unlike Cantor's procedure, it remains valid even in a "formalistic" axiomatized set theory.[16]

von Neumann -- the first to base the definition of cardinals on that of ordinals -- takes to be the basis of his exposition on ordinals that, 'every ordinal is the set of ordinals that precede it'. Consequently, 0=0, 1=(0), 2=(0,(0)) and so on. Furthermore, von Neumann will consider as 'givens' only the concepts of 'well ordered set' and 'similarity'. And it is on this basis that he proceeds, in his attempt to make Cantorian theory more concrete, i.e. more formalized.

As in the case of formalizing cardinality through the notion of equivalence, the concept of ordinality has been given, by Cantor and later mathematicians, a more formal explication through the notion of similarity. Two sets A and B are said to be similar (denoted $A \simeq B$) if there exists a one-to-one correspondence between them which retains the order -- i.e. $a \prec a_i$ if and only if $b \prec b_i$, where b and b_i are the corresponding elements to a and a_i, respectively. This correspondence between similar sets, Cantor referred to as an 'imaging'

---

[16] J. von Neumann, On the Introduction of Transfinite Numbers, from J. Van-Heijenhoort, "From Frege to Godel", Harvard University Press; p.347.

of these sets upon one another.[17] The sets A and B are said to be equal when they contain identical elements and the same ordering relation. The later is an important proviso, for infinite sets in particular, since there can be any number of ordering relations on the same set -- hence, for example, the sets $\{1,2,3,....\}$ ordered according to increasing magnitude and $\{....,3,2,1\}$ ordered according to decreasing magnitude cannot be considered equal.

It is obvious from the definition of similarity that every set is similar to itself, since one can define an identity mapping which will map every element of the set onto itself and where the order remains unchanged. It can also be shown that the similarity of sets is a transitive relation. Another very significant characteristic of similarity is that two ordered sets A and B have the same ordinal type if and only if they are similar. The relation of similarity to ordinality is analogous to the relation of equivalence to cardinality -- but whereas similarity will imply equivalence, by definition, in general the reverse is not usually the case, for it may be that two equivalent sets need not be ordered at all, hence any questions of similarity cannot enter in.

In the case of finite sets, the ordinal type is subject to the same arithmetic operation as the finite cardinal numbers.

Furthermore,

> Every type α has a definite cardinal number a. The various types of cardinality a form a class of types T [a]; to obtain this class one need only order a fixed set A of cardinality a in all the ways possible, where the different orderings of course do not necessarily yield distinct types.[18]

---

[17]Cantor's usage of the term 'imaging' (Abbildung) to denote similarity between two sets, is highly Platonistic. The term Abbildung (translated as copying, sketch, image, illustration) has obvious visual connotations. 'Visualize' two sets having a one-to-one correspondence between their respective elements. If we consider the 'imaging' as a process that we can 'see' as a matching of the elements. The term 'imaging' can be understood as a metaphor: In a (Platonistic) sense we can picture the process, or matching, taking place between two similar sets that 'exist'.

[18]Felix Hausdorff, Set Theory, Chelsea Pub., N.Y., 1957; p.57.

For a finite cardinal number a, all simply ordered sets of that cardinality are similar to one another and, consequently, have the same ordinal type. This does not occur, however, in the case of the transfinite ordinal -- for, to a particular cardinal number of an infinite set there can correspond an infinite number of types of simply ordered sets. And it is because of this attribute that Cantor has grouped all these 'types' into a 'class of types', as Hausdorff speaks of above, corresponding to a given transfinite cardinal number. That is, each of these classes is determined by the given transfinite cardinal. And because there are an infinite number of transfinite cardinals, there are, subsequently, an infinite number of classes of types. Although the concept of 'class of types' was never developed to any degree by the set theoreticians, it does serve to illustrate the fact that, in the realm of the transfinite, a number of new conditions arise with respect to the cardinal and ordinal number that are not present in the case of the finite; a fact which becomes even more apparent as one proceeds deeper in transfinite number theory. Another thing which these new conditions illustrate is the necessity of creating new conventions in order to develop the concepts of transfinite cardinal and ordinal numbers, since many of the old conventional ways of viewing and operating with finite cardinal and ordinal numbers are unsatisfactory 'beyond the finite'.

That is not to say, however, that mathematicians resort to some of the old habitual 'ways of thinking' when they sought to understand and theorize about the cardinal and ordinal numbers. Russell even goes so far as to 'prove' the existence of ω, the ordinal type of the positive integers ordered according to magnitude, and the rest of the transfinite numbers.

> The existence of ω (in the mathematical sense of existence) is not open to question since ω is the type of order presented by the natural numbers themselves. To deny ω would be to affirm that there is a last finite number -- a view which, as we have seen, leads at once to definite contradictions. And when this is admitted, $\omega + 1$ is the type of the series of ordinals including ω, i.e. of the series whose terms are all series of integers 1 up to any finite number together with the whole

> series of integers. Hence all the infinite hierarchy of transfinite numbers easily follows.[19]

That Russell deemed it a necessary step to give a 'proof' for the existence of the 'new numbers', is indicative of the existence convention which was spoken of earlier. This is obviously a superfluous move, from the standpoint of the development of the transfinite number theory. And it certainly does not change the operations and/or development relating to the concept of ordinality or infinite sets -- although it may affect their relative status, insofar as mathematical tradition is to be considered.

Although Cantor, like Russell above, did become concerned with the existence convention and some other long-standing conventional ways of viewing mathematics in general and the infinite in particular -- he did offer a new direction. What can be seen in Cantor's method of introducing both the cardinality and ordinality of a set, is the presence of a new language, new symbols, and a new approach in talking about the infinite. One may have already been able to detect a certain structure emerging within and about the transfinite number theory. New conventions are being introduced and developed for even the most basic concepts -- such as 'set', 'infinite', 'finite', 'cardinality', 'ordinality', -- of the theory. And as it is developed, one can see that there are parallel conventions and characteristics between certain aspects of the theory. (For instance, analogous to the cardinal number and equivalence of sets, is the notion of ordinal type and similarity of sets). And there are the interconnections, the creation of conventions and utilization of old conventions which are based on and/or expanded from the previous ones -- deriving from, in the final analysis, the basic set-theoretical concepts and both new and old operations, symbols, and 'thinking habits' about the theory.

The fact that so much of set theory was expanded from and used these basic concepts could be seen as one of the major drawbacks of the theory, for when contradictions began to appear and became known through-

---

[19] Bertrand Russell, <u>Principles of Mathematics</u>, George Allen and Unwin, 1903; p.362.

out the mathematical realm, the validity and value of the entire transfinite number theory was challenged. There were three important antinomies -- Russell's, Burali-Forti's and Cantor's -- which involved the primary concepts of 'cardinality', 'ordinality', and 'set'. And because of the strength of mathematical tradition regarding the presence of contradiction in any theory and the fact that such contradictions drew upon the foundation of the theory, itself, for the first time an entire mathematical creation was called into question.

The Russell contradiction, or antinomy, is perhaps the most widely known and certainly one of the most significant, since it involves only the notion of 'set':

> The comprehensive class we are considering, which is to embrace everything, must embrace itself as one of its members. In other words, if there is such a thing as 'everything', then 'everything' is something, and is a member of the class "everything". But normally a class is not a member of itself. Mankind, for example, is not a man. Form now the assemblage of all classes which are not members of themselves. This is a class: is it a member of itself or not? If it is, it is one of those classes that are not members of themselves, i.e. it is not a member of itself. Thus of the two hypotheses -- that it is, and that it is not, a member of itself -- each implies its contradictory. This is a contradiction.[20]

Russell suggests the fallacy to be in the 'impurity' of the definition, in the respect of its lack of restriction in clarifying what can be included under the concept. There must be a supplementation of further criteria whereby one can determine when a collection of objects, such as the 'set of all sets', can constitute a set.

Zermelo, who had also discovered this paradox, independently from Russell, presented the first axiomatization of the set theory. Whereas Russell had responded to the contradictions with the creation of a 'theory of types', Zermelo responded with the far-reaching axiomatization. In particular, his restriction on the concept of 'set', which would rule out 'sets' which are too large, was accomplished principally with his 'Axiom of Separation', which contained a number

---

[20] B. Russell, Introduction to Mathematical Philosophy, George Allen and Unwin Ltd., N.Y., 1919; p.136.

of restrictions:

> In the first place, sets may never be independently defined by means of this axiom but must always be separated as subsets from sets already given; thus contradictory notions such as 'the set of all sets' or 'the set of all ordinal numbers' . . . are excluded. In the second place, moreover, the defining criterion must always be definite in the sense of our definition in No.4 (that is, for each single element x of M the fundamental relations of the domain must determine whether it holds or not) . . . [21]

But conventions clashed with conventions: Conventionally, a contradiction such as that of Russell-Zermelo's brings the whole theory into doubt, or at least those parts (such as the basic concepts of the theory) which utilize anything which the contradiction might have involved. However, there was a conflict of sorts with the mathematical convention regarding 'existence and reality' of mathematical entities. Namely, there were some mathematicians (Fraenkel, for one) who felt the Russellian contradiction to be undercut by its presupposition that there actually would exist anything such as the 'set of all sets that are not members of themselves'. There would only be a paradox if such a set actually existed. This only slightly alleviated the impact of the contradiction, so far as most of the mathematicians indicated. Conventions regarding contradiction in a mathematical creation were extensive enough that even the possibility of a contradiction, no matter how slight or exaggerrated, (which is not to say that Russell's was) warrants a rather ominous shadow upon the theory.

Furthermore, such attempts as axiomatization to rehabilitate set theory by restricting the notion of set do not necessarily guarantee a way out of the difficulties. As von Neumann suggested:

> . . . [with] the axiomatic method . . . one formulates a number of postulates in which, to be sure, the word 'set' occurs without any meaning. Here . . . one understands by 'set' nothing but an object of which one knows no more and wants to know no more than what follows about it from the postulates. The postulates are to be formulated in

---

[21] E. Zermelo, Investigations in the Foundations of Set Theory I, from J. van-Heijenhoort, From Frege to Godel, Harvard University Press, 1967; p.202.

> such a way that all the desired theorems of Cantor's set theory follow from them, but not the antinomies. In these axiomatizations, however, we can never be sure of the latter point. We see only that the known modes of inference leading to the antinomies fail but who knows whether there are not others?[22]

It may very well be the case, therefore, that in no way can we rest assured that we have before us a 'contradiction-less' theory regarding transfinite numbers. Rather, one may either have to acquire a somewhat stoical attitude and accept the presence of possibility of paradoxes in the theory or to be continually suspicious and leery of the entire theory as being 'acceptable' mathematics.

The Burali-Forti antinomy contributed to the suspicions. The paradox can be stated simply, and on the basis of naïve concepts of the set theory: Because the set of ordinal numbers is itself well-ordered, it therefore has an ordinal number. This ordinal is both an element of the set (since it is an ordinal number) and is greater than any ordinal number in the set.

Cantor was also aware of this antinomy as early as 1895, and although he contended:

> What Burali-Forti has produced is thoroughly foolish. If you go back to his articles in Circolo Matematico, you will remark that he has not even understood properly the concept of a well-ordered set.[23]

Cantor responded by reconsidering the concept of 'set', attempting to place certain restrictions as to its generality. In his letter of 1899 to Dedekind, Cantor created a new convention of considering what he termed 'multiplicities' -- that is, systems, collections, totalities -- making a distinction as to what is 'consistent' or 'inconsistent'.

> . . . a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I

---

[22]John von Neumann, An Axiomatization of Set Theory, from J. van-Heijenhoort, From Frege to Godel, Harvard University Press; p.395.

[23]G. Cantor, as quoted by Hao Wang, Process and Existence in Mathematics, from Bar-Hillel, Poznanski, Rabin & Robinson (eds.), Essays on the Foundations of Mathematics, North-Holland, 1962; p.346.

> call absolutely infinite or inconsistent multiplicities . . . the 'totality of everything thinkable', for example, is such a multiplicity . . . . If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity of a 'set'. . . . Two equivalent multiplicities are both 'sets' or are both inconsistent.[24]

Cantor's distinction, though anticipatory of the set/class distinction which von Neumann was to make at a later date, is really somewhat ambiguous, for it does not offer much of a guideline or procedural rule for determining whether something is to be considered consistent or inconsistent. That is, one would have to know beforehand if the 'multiplicity' would or would not lead to a contradiction and only then could one be assured in calling the multiplicity a 'set'. More importantly than its ability to assist in making the definition of 'set' more precise, Cantor's distinction is more an acknowledgement of the mathematical disdain for contradiction. And it was strengthened by his discovery of yet another antinomy involving the basic concepts of his theory.

In 1899 Cantor discovered a paradox involving his notions of 'set' and 'cardinality'. It was derived in the following manner: Consider the set A of all sets. Obviously its cardinal number $\bar{\bar{A}}$ is the largest which can exist. On the other hand, by Cantor's theorem the 'power set' (set of all subsets of A) of the set has more elements and hence a greater cardinality than the set itself. Therefore the cardinal of the power set of A is greater than $\bar{\bar{A}}$, the cardinal of A.

The set theoretical antinomies, such as the three mentioned, were met by mathematicians with every type of response from indifference to fascination to alarm. There were mathematicians like Zermelo, von Neumann and Fraenkel, who attempted to 'patch up' set theory via the imposition of axioms, restrictions, etc. There were mathematicians

---

[24] G. Cantor, Letter to Dedekind, from J. van-Heijenhoort, From Frege to Godel, Harvard University Press, 1967; p.114.

who followed Frege in doubting whether arithmetic could possibly be given any kind of a reliable foundation. There were mathematicians who acted with acute uneasiness:

> In particular, a contradiction discovered by Zermelo and Russell had a downright catastrophic effect when it became known throughout the world of mathematics. Confronted by these paradoxes, Dedekind and Frege completely abandoned their point of view and retreated. Dedekind hesitated a long time before permitting a new edition of his epoch-making treatise 'Was sind und was sollen die Zahlen' to be published. In an epilogue, Frege too had to acknowledge that the direction of his book 'Grundgesetze der Mathematik' was wrong.[25]

There were mathematicians who were more reactionary and questioned the entire worth of the theory. Of these, many followed Poincaré in denying Cantor's theory of transfinite numbers the right to exist, proclaiming it a 'beautiful pathalogical case'. Thus were the varying reactions of mathematicians to, in Wittgenstein's words, the 'metaphysical thorn' which the presence of contradictions was seen to be in the side of set theory.

Wang questions why a proof of contradiction could not be treated as any other piece of mathematics, even though it may not be quite as significant as an 'ordinary' theorem. He further wonders why the discovery of inability (contradiction) of dividing by zero is a problem much easier disposed with than the contradictions in set theory. In recognition of the mathematical conventions and 'thinking habits' regarding contradiction, Wang notes:

> The usual reaction to the discovery of a contradiction is to analyze the moves involved in the derivation and pronounce some of the moves unwarranted. The repercussions of a contradiction include the rejection of all proofs which involve similar moves. In this sense, contradictions are contagious. Proofs which were otherwise considered healthy are put into concentrated isolation on account of their contact with contradictions.[26]

---

[25]D. Hilbert, On the Infinite, from Benacerraf & Putman (eds.) Philosophy of Mathematics, Prentice-Hall Inc., New Jersey, 1964; p.141.

[26]Hao Wang, Process and Existence in Mathematics, from Bar-Hillel, Poznanski, Rabin, & Robinson (eds.), Essays on the Foundations of Mathematics, North-Holland Puc., Amsterdam, 1962; p.344.

So it was, with the presence of contradictions that drew from the basic concepts of the theory, that the once 'healthy', though controversial Cantorian set theory came to be considered 'diseased'. And consequently:

> . . . one might be persuaded to leave the paradise into which Cantor has led the mathematicians and to withdraw into a less opulant but more secure habitat. Those unwilling to do this might perhaps prefer to stay within the realm of plenty and build walls around it to keep away the beastly antinomies without, however, being certain that some of these beasts were not walled in themselves.[27]

Let us return to an objective which was raised earlier about (clarifying) the difference between the 'potential' and 'actual' infinite. One can see -- presupposing that one has understood the basic set theoretical terms as Cantor introduced them and how they were altered and/or expanded upon by later mathematicians -- some general contrasts between the two approaches.

On the one hand, the 'potential' infinite approach can be considered rather constructionistic, insofar as the concept tends to be viewed more as a rule or process of generation, which continues indefinitely (the 'ever-expanding large', so to speak). This type of position is supported by traditional views as to what mathematics consists of and what types of operations are permissible in the realm of the infinite. In support of the standpoint that we can only really understand an 'infinite' which can be constructed, there are statements such as those contending that the finite intellect cannot grasp anything greater or beyond a potential infinite. But since such an infinite is never attainable (due to the fact there is no completion), such an 'infinity' can never be constructed and, therefore, we can never comprehend the infinite qua infinite.

On the other hand, there is the approach which Cantor took about the 'actual' infinite, and which hopefully has been made more clear by the examination of the primary concepts of his theory.

---

[27]A. Fraenkel, Foundations of Set Theory, North-Holland Publishers, Amsterdam, 1958; pp.13-14.

This approach differs considerably from the 'potential' infinity standpoint. Cantor's approach of 'going beyond the finite' might also be viewed as an approach of 'going beyond the (constructed) infinite', insofar as the 'potential' infinite could be seen as a kind of subsidiary of the 'actual' infinite. By this is meant that we can view, for instance, the number system as an endless series of ever increasing and decreasing terms and, therefore, the idea of a 'potential' infinite is already present. But taking a step above that and considering the totality (of which the 'potential' infinite is included, as something which the numbers are going toward) then we gain a sense of the 'actual' infinite. Namely, it (the 'actual' infinite) can be understood as a collection of psychological (Platonistic) entities rather than a mere 'figure of speech' or rule of generation which, since there is no completion,has only a symbolic finish. (It could be said, that the 'potential infinite' is analogous to 'Santa Claus' in that we may speak about both with the assumption that the listener has an understanding of the concept in question, but the speaker and the listener both know that such an entity has no 'real' existence, but only a symbolic/mythical one).

If one re-examines the two approaches on 'infinity' one might see that, in reality, the positions are not conflicting but, rather, are just very different perspectives. Where the conflict might come to fore is in the permissibility of taking the perspective in the first place. Cantor had introduced a direction which -- though influenced by several mathematical and philosophical modes of thinking -- had never been considered possible or plausible (and, consequently, had never been developed previously) in a mathematical theory.

Certainly Cantor's approach was new insofar as it offered a totally different perspective on the transfinite and introduced many concepts (e.g. 'set', 'cardinality', 'ordinality', similarity, one-to-one correspondence, and so on) which had not hitherto been considered and/or developed to any degree previously. Even in light of the Platonistic aura around Cantorian set theory, it still can be

considered a new direction, since never before had Platonism been utilized so extensively in a mathematical theory and certainly not with respect to 'infinity'. For instance, contrast views of Aristotle, Gauss, Kronecker, etc. where it is spoken of at most in only semi-Platonistic terms, for their usage of the infinite is as a symbol. Such mathematicians saw it only within a constructionistic framework -- where, as Aristotle contended, the 'mathematicians have no use of [the infinite], but only postulate that it become as long as they wish'. Consequently, even though Platonism has long existed within mathematics, it had not really reached its climax, so to speak, prior to Cantor. And although Cantor employed it (Platonism) to considerable lengths -- that alone was new -- Cantor's transfinite number theory must be seen as a new approach, a 'new way of seeing' the infinite.

CONCLUSION

It has been said that Cantorian set theory precipitated one of the biggest disputes in history as to the validity of (traditional) mathematical reasoning. However, it seems that the conflict, if not non-existent, is misdirected and out of proportion. That is, there is no real opposition between the two viewpoints of 'infinity', because they constitute two different directions altogether.

The actual/potential infinity dispute encompassed some of the most basic conventions in mathematical thought. Cantor seemed fully aware of the old conventions regarding the 'potential' infinite, but felt that he was unable to continue his theory on the basis of the old structure; namely, the constructionistic view of the 'potential' infinite. But the traditional modes of thinking on the subject were so strong that Cantor stated:

> I depend upon that generalization of number concept to such an extent that without it I should hardly be able freely to take even the least step forward in the theory of sets; may this serve as a justification or, if necessary, an apology for my introducing seeming strange ideas into my considerations.[1]

As mentioned earlier, the strength of conventions creates a kind of inhibitory reasoning which makes for hesitancy (obvious in Cantor's case) in introducing a new concept(s) or theory into mathematics.

Cantor offered a new perspective of the infinite which was to be more than merely an alternative to the traditional view. That is, the issue of the 'potential'/'actual' infinite did not reduce itself just to a matter of convention, although people reacted to Cantor's writings as if he were trying to replace the old system. What he was offering, rather, was an approach which was reconciliable (mathematically -- philosophically there may be some dispute) with the old one. And, whereas it was seen as a threatening attempt to refute the old theories, Cantor's creation was essentially a separate consideration altogether. Of course, this is not to say that Cantor

---

[1] G. Cantor, Introduction to the Founding of the Theory of Transfinite Numbers, Dover Publications, N.Y., 1915; p.53.

Note: The emphasis is my own.

himself did not get caught up with mathematical tradition and 'habits of thought'. Quite the opposite.

Throughout this paper some major conventions within mathematics have been discussed, particularly regarding 'infinity' and the perspectives taken upon the notion of (actual) infinite sets. One convention which stands out is that of 'existence and reality', which demands a mathematical entity of some abstract or concrete form to provide a referrent for the mathematical theory in question. (So, in this way mathematics could not really be viewed as mere manipulation of symbols, as some formalists might suggest).

The 'mathematics as descriptive' convention became a significant consideration in Cantor's time, specifically since mathematics was then more materially oriented, or 'applied'. It continued to be a concern when 'pure' mathematics developed. Even there it was expected that the theory have a referrent, although a physical one usually could not be found (e.g. the case of transfinite number theory).

When no physical correlation was found for Cantorian set theory and when it became further obvious that there were no acceptable 'proofs' for the existence of infinite sets, mathematicians (grudgingly?) settled upon the Zermelo-Fraenkel Axiom of Infinity. As we have seen, Cantor himself became involved with this whole issue of showing 'existence'. He not only attempted to justify such an introduction in the first place, but also sought to demonstrate the actuality of the infinite sets and the transfinite numbers. (For instance, see his attempts to show that the transfinite numbers 'take a perfectly determined and distinguishable place in our understanding').

Cantor's attempts and those of later mathematicians, with their postulated infinity of sets, demonstrates the powerful 'ways of thinking' and the fact that habit can become deadening. So much so, that the 'proofs' and postulates to provide existence of mathematical entities still go unquestioned. They still have yet to be judged the irrelevancies which they are -- particularly in the case of 'pure' mathematics, where the issue of application is secondary.

A part of the 'mathematics as descriptive' convention -- a sub-

set, metaphorically speaking -- is a way of thinking in both mathematics and philosophy that Cantor was obviously affected by and, in fact, expanded upon to a great extent. This is Platonism. "Absolute Platonism", as Bernays suggests, postulates the existence of a world of ideal objects containing all the objects and relations of mathematics.[2] As was shown in the last chapter, via an exposition of Cantor's definitions of the basic terms in this theory, Cantor developed a strong Platonism. This extended throughout his theory and, thus, obviously surrounded the concept of the 'actual' infinite. To substantiate this claim, one need only re-examine the concept of set, which includes anything capable of being an object of "our intuition or our thought" and his concepts of ordinal type and cardinal number, which embrace "everything capable of being numbered that is thinkable". Since Cantor's notion of the 'actual' infinite deals with the totality of all that is thinkable and treats the totality itself as a mathematical entity, it seems that Cantor has definite Platonistic leanings.

To further support this claim and, consequently, Cantor's tie to a very powerful 'habit of thinking' within the 'mathematics as descriptive' convention, let us look at two statements that Cantor made. Writing about the mathematician Veronese, Cantor said:

> . . . we ought not to be surprised at the lawlessness with which later on he operates with his <u>pseudo-transfinite numbers</u>, and ascribes properties to them which they cannot possess simply because they themselves, in the form imagined by him, <u>have no existence except on paper</u>.[3]

And, in an exposition of the concept of cardinality, Cantor stated:

> Since every single element m, if we <u>abstract from its nature</u>, becomes a 'unit' the cardinal number M is a definite aggregate composed of units, and <u>this number</u>

---

[2]See Bernays' essay, "On Platonism" where he speaks about the Platonism throughout Cantorian set theory arguing that the position of 'absolute platonism' has been shown untenable by the set-theoretical antinomies, particularly that of Zermelo-Russell.

[3]G. Cantor, <u>Contribution to a Founding of a Theory of Transfinite Numbers</u>, Dover Publications, N.Y., 1915; p.118.
<u>Note</u>: The emphasis is my own.

has existence in our mind as an intellectual image or projection of the given aggregate M.[4]

As in Cantor's 'acts of abstraction', to obtain the cardinal and ordinal numbers of a set, this presents a strongly Platonistic view toward mathematical entities. Like Plato's 'ideal objects' which have only to be discovered to be known (in contrast to being created) -- it seems that mathematical entities according to Cantor's usage are things which we utilize, abstract from and 'act upon'. For instance, in the case of the 'acts of abstraction', the entities get transformed by certain operations into ordinal numbers. Another operation or 'act of abstraction' will put the object through another transformation, so that it then 'turns into' a cardinal number. What seems to be presupposed is that these entities have a very real kind of existence from the beginning -- so that both the ordinal and cardinal numbers result from an action taken upon an already existent entity. This is also demonstrated by Cantor's employment of the phrase, 'abstract from its nature' and the phrase, 'intellectual image. . . of the given set M'.

The above remarks should also give some insight into Cantor's condemnation of Veronese. In referring to the 'pseudo-transfinite' numbers, with 'no existence except on paper', Cantor suggests that these entities do not really exist, implying that even to utilize them is 'lawless', or criminal. And since they only 'exist' on paper, Veronese is incorrect in ascribing them properties. It is important to note Cantor's implication that the existence of a mathematical entity is a necessary condition for that entity to possess any kind of properties whatsoever. But perhaps this can provide us with more understanding of Cantor's justification of what he had created. After the remarks made about Veronese, to maintain good faith Cantor can ascribe characteristics to his transfinite numbers only after they themselves can be shown to have a real existence -- hence, his concern with the 'existence and reality' convention.

---

[4] Ibid., p.86. Again: my emphasis.

That Cantor had been influenced by 'habits of thinking' should be apparent. It can be seen and has been pointed out by current mathematicians (e.g. Skolem, Bernays) the extent to which the transfinite number theory is Platonistic. The Platonistic kind of thinking of Cantor's -- in viewing mathematical entities as psychological entities -- is in quite a contrast to the traditional view of the 'potential' infinite, a constructionistic approach. And, in light of Cantor's emphasis on the 'completed', 'actual' infinite, which is of far more Platonistic leanings, the response from those like Kronecker and Brouwer, with more constructionistic (intuitionistic) tendencies should not be surprising.

The extent to which Cantor did get involved (whether consciously or from 'habit') with the old conventions of mathematics, should demonstrate the strength of tradition within the mathematical world. Indeed, it may be asked, did Cantor, in fact, offer a 'new way of seeing', as he contended? Is Cantor's having been affected by the traditional modes of thinking in mathematics to be considered in any way a refutation that he failed to create a new approach to the concept of 'infinity'?

BIBLIOGRAPHY

Anderson, Alan Ross. "Mathematics and the Language Game", from Benacerraf and Putnam, Philosophy of Mathematics. Prentice-Hall, Inc., New Jersey, 1964.

Aposte, Hippocrates George. Aristotle's Philosophy of Mathematics. University of Chicago Press, 1952.

Bell, E.T. Men of Mathematics. Simon and Schuster, New York, 1937.

Bar-Hillel, Yehosha. Mathematical Logic and Foundations of Set Theory. North-Holland Publishers, Amsterdam, 1968.

Beth, Evert W. The Foundations of Mathematics. Harper and Row, New Jersey, 1959

__________. Mathematical Thought. D. Reidel Publishing Company, New York, 1965.

Bernays, Paul. "On Platonism", from Benacerraf and Putnam (ed.), Philosophy of Mathematics. Prentice-Hall, New Jersey, 1964.

Bochenski, I.M. Ancient Formal Logic. North-Holland Publishing Company, Amsterdam, 1968.

Bowne, G.D. The Philosophy of Logic, 1880-1908. Mouton and Company, London, 1966.

Breur, Joseph. Introduction to the Theory of Sets. Prentice-Hall Inc., New Jersey, 1958.

Burali-Forti, Cesare. "A Question on Transfinite Numbers" (1897), from Jean Van-Heijenhoort, From Frege to Godel. Harvard University Press, 1967.

Cantor, Georg. "Letter to Dedekind (1899)", from Jean Van-Heijenhoort, From Frege to Godel, Harvard University Press, 1967.

__________. Introduction to the Founding of a Theory of Transfinite Numbers. Translated by Phillip Joudain. Dover Publications, New York, 1915.

Carrucio, Ettore. Mathematics and Logic in History and in Contemporary Thought. Faber and Faber, London, 1964.

Curry, Haskell. Outlines of a Formalist Philosophy of Mathematics. North-Holland Publishers, Amsterdam, 1951.

Dedekind, Richard. Essays on the Theory of Numbers. Open Court Publishing Company, Illinois, 1948.

Eves, Howard. Introduction to the History of Mathematics. Holt Rinehart and Winston, New York, 1964.

Fraenkel, Abraham. Abstract Set Theory. North-Holland Publishers, Amsterdam, 1961.

Fraenkel, Abraham and Y. Bar-Hillel. Foundations of Set Theory. North-Holland Publishers, Amsterdam, 1958.

Frege, Gottleib. Foundations of Arithmetic. Basil Blackwell, Oxford, 1959.

Gödel, Kurt. "What is Cantor's Continuum Problem?" From Benacerraf and Putnam (eds.) Philosophy of Mathematics. Prentice-Hall Inc., New Jersey, 1964.

Goodstein, R.L. Constructive Formalism. University Press, Leicester, England, 1965.

Hausdorrf, Felix. Set Theory. Chelsea Publishers, New York, 1957.

Heyting, A. "Axiomatic Method and Intuitionism", from Bar-Hillel, Pznanski, Rabin (eds.), Essays on the Foundations of Mathematics. North-Holland Publishers, Amsterdam, 1962.

Hilbert, D. "On the Infinite", from Benacerraf and Putnam (eds.), Philosophy of Mathematics. Prentice-Hall Inc., New Jersey, 1964.

Huntington, E.V. The Continuum and Other types of Serial Order with an Introduction to Cantor's Transfinite Numbers. Dover Publications, Massachusetts, 1917.

Kattsoff, Louis. A Philosophy of Mathematics. Iowa State College Press, 1948.

Keene, Geoffrey. Abstract Sets and Finite Ordinals. Pergamon Press, Oxford, 1961.

Kleene, Stephen Cole. The Foundations of Intuitionistic Mathematics. North-Holland Publishers, Amsterdam, 1965.

__________. Introduction to Metamathematics. D. van Nostrand Company, Inc., Princeton, New Jersey, 1953.

Kline, Morris. Mathematics in Western Culture. Oxford University Press, New York, 1953.

Korner, Steven. Philosophy of Mathematics. Harper and Row, New York, 1960.

Lakatos, Imre. Problems in the Philosophy of Mathematics. North-Holland Publishers, Amsterdam, 1967.

Manning, Henry F. Irrational Numbers and Their Representation by Sequences and Series. Wiley Publishers, New York, 1906.

Meschkowski, Herbert. Ways of Thought of Great Mathematicians. Holden-Day Company, London, 1964.

Nagel, Ernest. Logic Without Metaphysics. Free Press, Illinois, 1956.

Nagel, Ernest and M. Cohen. An Introduction to Logic and Scientific Method. Harcourt, Brace and World, Inc., New York, 1934.

Poincaré, Henri. Mathematics and Science: Last Essays. Dover Publications, New York, 1963.

Quine, Willard V. From a Logical Point of View. Harper and Row, New York, 1953.

__________. "Truth by Convention", from Benacerraf and Putnam (eds.), Philosophy of Mathematics. Prentice-Hall Inc., New Jersey, 1964.

__________. Mathematical Logic. Harper and Row, New York, 1950.

__________. Set Theory and Its Logic. The Belknap Press, Cambridge, Massachusetts, 1963.

__________. Philosophy of Logic. Prentice-Hall Inc., New Jersey, 1970.

Rottman, B. and G.T. Kneebone. The Theory of Sets and Transfinite Numbers. Elsevier Publishing Company, New York, 1966.

Russell, Bertrand. The Principles of Mathematics. George Allen and Unwin Limited, London, 1903.

__________. Mathematical Philosophy. MacMillan and Company, London, 1920.

Russell, Bertrand and A.N. Whitehead. Principia Mathematica. Cambridge University Press, London, 1967.

Sierpenski, Waclaw. Cardinal and Ordinal Numbers. Warsaw Polish Scientific Publishers, 1965.

__________. Introduction to the Theory of Sets and Topology. Indiana Press Printers Limited.

Skolem, Thoralf A. Abstract Set Theory, Notre Dame Mathematical Lectures, No.8, Notre Dame, 1962.

Stoll, Robert. Set Theory and Its Logic. W.H. Freeman and Company, San Francisco, 1963.

__________. Sets, Logic and Axiomatic Theories. W.H. Freeman and Company, San Francisco, 1961.

von Neumann, John. "An Axiomatization of Set Theory", from Jean van-Heijenhoort, From Frege to Godel. Harvard University Press, 1967.

__________. "On the Introduction of Transfinite Numbers (1923)", from Jean van-Heijenhoort, From Frege to Godel. Harvard University Press, 1967.

Wang, Hao. "Process and Existence in Mathematics" from Bar-Hillel, Poznanski, Rabin and Robinson (eds.), Essays on the Foundations of Mathematics. North-Holland Publishers, Amsterdam, 1962.

Wilder, J. Introduction to the Foundations of Mathematics. Wiley Publishers, New York, 1965.

Wittgenstein. Ludwig. The Blue and Brown Books. Harper and Row, New York, 1958.

__________. Remarks on the Foundations of Mathematics. M.I.T. Press, Massachusetts, 1967.

Zermelo, Ernst. "Investigations in the Foundations of Set Theory", from Jean van-Heijenhoort, From Frege to Godel. Harvard University Press, 1967.

__________. "A New Proof of the Possibility of a Well-Ordering", from Jean van-Heijenhoort, From Frege to Godel. Harvard University Press, 1967.